Square-well solution to the three-body problem

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The angular part of the Faddeev equations is solved analytically for s-states for two-body square-well potentials. The results are, still analytically, generalized to arbitrary short-range potentials for both small and large distances. We consider systems with three identical bosons, three non-identical particles and two identical spin-1/2 fermions plus a third particle with arbitrary spin. The angular wave functions are in general linear combinations of trigonometric and exponential functions. The Efimov conditions are obtained at large distances. General properties and applications to arbitrary potentials are discussed. Gaussian potentials are used for illustrations. The results are useful for numerical calculations, where for example large distances can be treated analytically and matched to the numerical solutions at smaller distances. The saving is substantial.
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I. INTRODUCTION

A new method to study the quantum mechanical three-body problem was recently formulated [1]. The first significant advantage is the precise treatment of large distances where the correct asymptotic behavior of the wave function is incorporated. Another advantage is the parallel treatment of bound and continuum states, which in combination with the long-range treatment opens the interesting possibility of approaching the celebrated three-body Coulomb problem in the continuum. The method solves the coordinate space Faddeev equations in two steps. First the angular (and most difficult) parts of the three-component wave functions are calculated by elaborate use of the analytical knowledge of the large-distance behavior. Then the coupled set of effective radial equations is solved numerically [2].

The new method is designed to solve the Faddeev equations in coordinate space with the advantage of an immediate intuitive interpretation of the physics involved. (It may be considered as a development from the approach used to study the properties of H^- [3].) The advantages are seen in the analytical and numerical treatment of extreme cases like the so-called Efimov states [1,4], which has been suggested and looked for in both molecules [5] and nuclei [6]. These states occur when at least two of the two-body subsystems simultaneously have an s-state at zero energy arising from short-range interactions. The resulting infinitely many bound three-body states of 0^+ nature are extremely extended in space and extremely weakly bound. The long range Coulomb interaction between only one pair of particles destroys the effect.

The large-distance coupling takes place only between s-states in different relative two-body subsystems [2]. Furthermore, low-lying three-body states often contain large components of such s-states. Different treatments are usually needed when long-range interactions like the Coulomb potential are involved and we shall only consider short-range interactions. We shall furthermore neglect the coupling to higher angular momenta and confine ourselves to relative s-states. The purpose of this paper is to classify the lowest eigenvalues and describe how the solutions for small as well as large distances can be obtained analytically for the angular part of the Faddeev equations. These solutions turn out to be exact for square-well potentials where we also can find exact solutions at intermediate distances. Thus we shall derive a semianalytic s-state square-well solution to the Faddeev equations.

The solutions describe the Efimov anomaly and how the infinitely many states continuously appear and disappear as function of the parameters of the potential. The power of the method is especially due to the analytical treatment of large distances which must be treated with particular care for loosely bound quantum systems in low angular momentum states. Also the connection between two- and three-body large-distance behavior can be explored. Using the analytical results for small and large distances in numerical calculations for arbitrary potentials improve both precision and computational speed and enable thereby investigations of otherwise inaccessible problems.

Other methods are available for investigations of the three-body problem for baryons [7], for Borromean systems [8,9], for molecular systems [10,11] as well as for Coulomb interacting particles [12,13]. The results from all these methods are needed in studies of the properties of a variety of different systems of interest in physics, see for example [14–17]. Some of the methods solve the Schrödinger equation directly, but the Faddeev equations are needed to describe all the subtle correlations. This is especially clearly seen in systems where two- and three-body asymptotic behavior are mixed [2]. Borromean systems, characterized by genuine three-body asymptotics only, are relatively easy to handle, whereas weakly bound systems, where both two and three-particle correlations are essential, require very careful treatment of the large distances.

Analytic results are rare in quantum mechanics where the Schrödinger equation should be solved. The Faddeev equations further complicate analytic analyses. However, so far at least one exception exists for identical spinless particles interacting via two-body harmonic oscillator potentials [18]. This potential is infinite at large distance and scattering states therefore cannot be studied. Also the behavior for short-range potentials in general are excluded.

In this paper we outline in section 2 the general theoretical framework and in section 3 we solve the angular eigenvalue problem for s-states for a system of three identical bosons and spin-independent interactions. In section 4 we generalize to systems of three different particles and in section 5 we consider a system with two identical spin-1/2 particles plus a third particle. In section 6 we give numerical illustrations and indicate qualitatively how to generalize the results. Tedious mathematical derivations are collected in appendices. Finally we give a summary and the conclusions in section 7.

II. THEORETICAL FRAMEWORK

The intrinsic Hamiltonian of the three-body system is given by

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m_i} - \frac{P^2}{2M} + \sum_{i>j=1}^{3} V_{ij} \ , \tag{1}$$

where m_i, $\mathbf{r}_i$ and $\mathbf{p}_i$ are mass, coordinate and momentum of the i'th particle, V_{ij} are the two-body potentials, P and M are respectively the total momentum and the total mass of the system. We shall use the (three sets of) hyperspherical coordinates which consist of one radial coordinate ρ (hyperradius) and five generalized angles Ω_i, where i= 1, 2, 3. The precise definitions are given in appendix A. One of these sets of hyperspherical coordinates is in principle sufficient for a complete description. The volume element is given by $\rho^5 \mathrm{d}\Omega \mathrm{d}\rho$ where $\mathrm{d}\Omega = \sin^2\alpha \cos^2\alpha \mathrm{d}\alpha \mathrm{d}\Omega_x \mathrm{d}\Omega_y$.

A. General procedure

The total wave function Ψ of the three-body system is written as a sum of three components $\psi^{(i)}$ which in turn for each ρ are expanded in a complete set of generalized angular functions:

$$\Psi = \sum_{i=1}^{3} \psi^{(i)} = \frac{1}{\rho^{5/2}} \sum_{i,n} f_n(\rho) \Phi_n^{(i)}(\rho, \Omega_i) \ , \tag{2}$$

where the radial expansion coefficients $f_n(\rho)$ are component independent and $\rho^{-5/2}$ is the phase-space factor. These wave functions satisfy the three Faddeev equations [19]

$$(T - E)\psi^{(i)} + V_{jk}(\psi^{(i)} + \psi^{(j)} + \psi^{(k)}) = 0 \ , \tag{3}$$

where E is the total energy, T is the kinetic energy operator and $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. The Faddeev equations may have non-trivial spurious solutions, where each component is non-vanishing while the sum corresponding to the Schrödinger wave function is identically equal to zero. The components of such solutions are eigenfunctions of the kinetic energy operator with eigenvalues equal to the total energy.

The procedure is now for each ρ to solve the eigenvalue problem for the five dimensional angular part of the Faddeev operator:

$$\frac{\hbar^2}{2m} \frac{1}{\rho^2} \hat{\Lambda}^2 \Phi_n^{(i)} + V_{jk}(\Phi_n^{(i)} + \Phi_n^{(j)} + \Phi_n^{(k)}) = \frac{\hbar^2}{2m} \frac{1}{\rho^2} \lambda_n(\rho) \Phi_n^{(i)} \ , \tag{4}$$

where $\hat{\Lambda}^2$ is the ρ-independent part of the kinetic energy operator defined by

$$T \equiv T_\rho + \frac{\hbar^2}{2m}\frac{1}{\rho^2}\hat{\Lambda}^2,\ T_\rho = -\frac{\hbar^2}{2m}\left(\rho^{-5/2}\frac{\partial^2}{\partial\rho^2}\rho^{5/2} - \frac{1}{\rho^2}\frac{15}{4}\right)\ . \tag{5}$$

Explicitly the generalized angular momentum operator $\hat{\Lambda}^2$ is given by

$$\hat{\Lambda}^2 = -\frac{1}{\sin\alpha\cos\alpha}\frac{\partial^2}{\partial\alpha^2}\sin\alpha\cos\alpha + \frac{\hat{l}_x^2}{\sin^2\alpha} + \frac{\hat{l}_y^2}{\cos^2\alpha} - 4 \tag{6}$$

in terms of an arbitrary α-coordinate and the angular momentum operators $\hat{l}_x^2$ and $\hat{l}_y^2$ related to the Jacobi coordinates. The spurious states are characterized by angular eigenvalues equal to those of the angular kinetic energy operator $\hat{\Lambda}^2$, i.e. $K(K+4), K = 0, 1, 2...$

Insertion of $\psi^{(i)}$ defined in eq. (2) into eq. (3) then leads to the coupled set of "radial" differential equations

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{2mE}{\hbar^2} + \frac{1}{\rho^2}\left(\lambda_n(\rho) + \frac{15}{4}\right)\right) f_n = \sum_{n'}\left(2P_{nn'}\frac{\mathrm{d}}{\mathrm{d}\rho} + Q_{nn'}\right) f_{n'}\ , \tag{7}$$

with the functions P and Q defined by

$$P_{nn'}(\rho) \equiv \sum_{i,j=1}^{3}\int d\Omega \Phi_n^{(i)*}(\rho,\Omega)\frac{\partial}{\partial\rho}\Phi_{n'}^{(j)}(\rho,\Omega), \tag{8}$$

$$Q_{nn'}(\rho) \equiv \sum_{i,j=1}^{3}\int d\Omega \Phi_n^{(i)*}(\rho,\Omega)\frac{\partial^2}{\partial\rho^2}\Phi_{n'}^{(j)}(\rho,\Omega)\ . \tag{9}$$

The diagonal part of the P-matrix vanishes, i.e. $P_{nn} = 0$.

B. Angular eigenvalue equation

It is convenient to show explicitly the spin dependence of the wave function $\Phi_n^{(i)}(\rho,\Omega_i)$ in eq. (2), see e.g. [19]. Let us consider s-waves only and assume that $\mathbf{s}$ is the intermediate spin obtained by coupling of the spins $\mathbf{s}_j$ and $\mathbf{s}_k$ of particles j and k. The spin $\mathbf{s}$ is afterwards coupled to $\mathbf{s}_i$ to give the total spin S of the system. The total angular wave function for the i'th channel then factorizes into the spatial part $\phi_{n,s}^{(i)}(\rho,\Omega_i)$ and the spin dependent part $\chi_{n,s}^{(i)}$, i.e.

$$\Phi_n^{(i)}(\rho,\Omega_i) = \frac{1}{\sin(2\alpha_i)}\sum_s \phi_{n,s}^{(i)}(\rho,\Omega_i)\chi_{n,s}^{(i)}\ , \tag{10}$$

where we explicitly extracted the phase-space factor $\sin(2\alpha_i)$. Both ϕ and χ may depend on the intermediate coupling.

The Faddeev components in eq. (3) must be expressed in one Jacobi coordinate set. For the s-waves the wave functions $\phi_{n,s}^{(k)}$, which only depend on α_k and ρ, are first expressed in terms of the i'th set of hyperspherical coordinates and subsequently integrated over the four angular variables describing the directions of $\mathbf{x}_k$ and $\mathbf{y}_k$. This amounts, still for s-waves, to the substitution formally expressed by the operator R_{ik} defined by

$$R_{ik}\left[\frac{\phi_{n,s}^{(k)}}{\sin(2\alpha_k)}\right] \equiv \frac{1}{\sin(2\varphi_j)}\frac{1}{\sin(2\alpha_i)}\int_{|\varphi_j-\alpha_i|}^{\pi/2-|\pi/2-\varphi_j-\alpha_i|}\phi_{n,s}^{(k)}(\rho,\alpha_k)d\alpha_k\ , \tag{11}$$

where the angle φ_j is given by the masses as

$$\tan\varphi_j = \sqrt{\frac{m_j(m_1+m_2+m_3)}{m_i m_k}}\ . \tag{12}$$

It is closely related to the transformation angle φ_{ik} used in appendix A and defined by

$$\varphi_{ik} = (-1)^p \varphi_j = \arctan\left((-1)^p \sqrt{\frac{m_j(m_1+m_2+m_3)}{m_i m_k}} \right) , \tag{13}$$

where $(-1)^p$ is the parity of the permutation $\{i,k,j\}$.

Substituting eq. (10) into eq. (4) we obtain, after multiplication from the left with $\chi^{(i)}_{n,s}$, the angular eigenvalue equation

$$\left(-\frac{\partial^2 \phi^{(i)}_{n,s}(\rho,\alpha_i)}{\partial \alpha_i^2} + (\rho^2 \langle \chi^{(i)}_{n,s} | v_i(\rho \sin\alpha_i) | \chi^{(i)}_{n,s} \rangle - \tilde{\lambda}_n(\rho)) \phi^{(i)}_{n,s}(\rho,\alpha_i) + \rho^2 \sin(2\alpha_i) \right.$$
$$\left. \times \sum_{s's''} \langle \chi^{(i)}_{n,s} | v_i(\rho\sin\alpha_i) | \chi^{(i)}_{n,s''} \rangle \left(C^{ij}_{s''s'} R_{ij} \left[\frac{\phi^{(j)}_{n,s'}}{\sin(2\alpha_j)} \right] + C^{ik}_{s''s'} R_{ik} \left[\frac{\phi^{(k)}_{n,s'}}{\sin(2\alpha_k)} \right] \right) \right) = 0 , \tag{14}$$

where $v_i(x) = V_{jk}(x/a_{jk}) 2m/\hbar^2$ with a_{jk} defined in appendix A, $\tilde{\lambda}_n(\rho) = \lambda_n(\rho) + 4$ and the coefficients $C^{ik}_{ss'}$ expressing the overlap of the spin functions are given by

$$C^{ik}_{ss'} = \langle \chi^{(i)}_{n,s} | \chi^{(k)}_{n,s'} \rangle . \tag{15}$$

These matrix elements are diagonal for $i = k$, i.e. $C^{ii}_{ss'} = \delta_{ss'}$ and symmetric, i.e. $C^{ik}_{ss'} = C^{ki}_{s's}$.

When the potential is diagonal in spin-space we obtain the much simpler set of equations

$$\left(-\frac{\partial^2 \phi^{(i)}_{n,s}(\rho,\alpha_i)}{\partial \alpha_i^2} + (\rho^2 \langle \chi^{(i)}_{n,s} | v_i(\rho \sin\alpha_i) | \chi^{(i)}_{n,s} \rangle - \tilde{\lambda}_n(\rho)) \phi^{(i)}_{n,s}(\rho,\alpha_i) + \rho^2 \sin(2\alpha_i) \right.$$
$$\left. \times \langle \chi^{(i)}_{n,s} | v_i(\rho\sin\alpha_i) | \chi^{(i)}_{n,s} \rangle \sum_{s'} \left(C^{ij}_{ss'} R_{ij} \left[\frac{\phi^{(j)}_{n,s'}}{\sin(2\alpha_j)} \right] + C^{ik}_{ss'} R_{ik} \left[\frac{\phi^{(k)}_{n,s'}}{\sin(2\alpha_k)} \right] \right) \right) = 0 . \tag{16}$$

Eqs. (14) and (16) constitute sets of equations obtained for i=1,2,3 and all possible (i-dependent) values of s. As usual the values of $\{i,j,k\}$ must here be a permutation of $\{1,2,3\}$.

C. Spin independent interactions

When the interactions are independent of spin, each Faddeev component must factorize into a spin-part and a spatial part. Furthermore, the spin can be factorized out of eq. (4) and the structure of the Faddeev equations then remains unchanged for the spatial parts alone. The spin dependent wave function must then be the same for all three Faddeev components and the spatial parts of the wave function in eq. (10) can at most differ by a normalization constant, i.e.

$$\phi^{(i)}_{n,s}(\rho,\Omega) \equiv b^{(i)}_s \phi^{(i)}_n(\rho,\Omega) , \tag{17}$$

$$\sum_s b^{(i)}_s \chi^{(i)}_{n,s} = \sum_s b^{(k)}_s \chi^{(k)}_{n,s} , \quad i,k = 1,2,3. \tag{18}$$

Multiplication of eq. (18) by $\chi^{(i)}_{n,s}$ from the left then gives

$$b^{(i)}_s = \sum_{s'} C^{ik}_{ss'} b^{(k)}_{s'} , \quad i,k = 1,2,3. \tag{19}$$

These equations are not independent, since

$$\sum_{s'} C^{ik}_{ss'} b^{(k)}_{s'} = \sum_{s'} C^{il}_{ss'} b^{(l)}_{s'} , \tag{20}$$

for all values of s, k and l. This is easily seen by use of the closure relation of eq. (15), i.e.

$$C^{ik}_{ss'} = \sum_{s''} C^{il}_{ss''} C^{lk}_{s''s'} \ , \ \ l = 1,2,3 \ . \tag{21}$$

Thus eq. (19) is valid for all k, if it only holds for one of the values of k. If $b_s^{(3)}$ is arbitrarily chosen and $b_s^{(1)}$ and $b_s^{(2)}$ calculated (i=1,2 and k=3) from eq. (19), we can see by using eqs. (21) and (20), that eq. (19) is valid for all other values of i and k. Thus any choice of $b_s^{(i)}$ for one value of i provides the same spin-independent solution of the Faddeev equations.

In the symmetric case when the three particles furthermore have equal masses and spatial interactions, the wave functions $\Phi_n^{(i)}$ corresponding to the different Faddeev components are independent of i. Then eq. (10) implies that

$$\sum_s \phi^{(i)}_{n,s}(\rho,\Omega)\chi^{(i)}_{n,s} = \sum_s \phi^{(k)}_{n,s}(\rho,\Omega)\chi^{(k)}_{n,s} \ , \tag{22}$$

which by use of eq. (17) directly leads to eq. (18)

Inserting eq. (17) into eq. (16) we obtain by use of eq. (19) the equation

$$\begin{aligned} &\left(-\frac{\partial^2 \phi_n(\rho,\alpha_i)}{\partial \alpha_i^2} + (\rho^2 v_i(\rho\sin\alpha_i) - \tilde{\lambda}_n(\rho))\phi_n(\rho,\alpha_i) \right. \\ &\left. + \rho^2 \sin(2\alpha_i) v_i(\rho\sin\alpha_i) \left(R_{ij}\left[\frac{\phi_n}{\sin(2\alpha_j)}\right] + R_{ik}\left[\frac{\phi_n}{\sin(2\alpha_k)}\right]\right)\right) = 0 \ , \end{aligned} \tag{23}$$

which determines $\phi_n \equiv \phi_n^{(i)}$ for i=1,2,3.

III. THREE IDENTICAL BOSONS FOR SPIN INDEPENDENT INTERACTIONS

In the symmetric case with spin independent interactions the remaining single Faddeev equation eq. (23) can by use of eqs. (11) and (12) explicitly be written as

$$\begin{aligned} &-\frac{\partial^2 \phi(\rho,\alpha)}{\partial \alpha^2} + (\rho^2 v(\rho\sin\alpha) - \tilde{\lambda}(\rho))\phi(\rho,\alpha) \\ &\qquad + \frac{4}{\sqrt{3}}\rho^2 v(\rho\sin\alpha) \int_{|\pi/3-\alpha|}^{\pi/2-|\pi/6-\alpha|} \phi(\rho,\alpha')d\alpha' = 0 \ , \end{aligned} \tag{24}$$

where we omitted the label "n". We shall only consider short-range potentials and often further restrict ourselves to square wells. It is most convenient first to solve exactly for the schematic square-well potential and afterwards generalize as much as possible. The two-body potential is then a step function

$$V(r) = -V_0 \Theta(r < R_0) \tag{25}$$

and the reduced potential in eq. (24) is therefore another stepfunction

$$v(x) = -v_0 \Theta(x < X) \ , \tag{26}$$

where $v_0 = 2mV_0/\hbar^2$ and $X = R_0/\sqrt{2}$ when the normalization mass is equal to the mass of one the particles. The angular Faddeev equation in eq. (24) is now solved analytically in different intervals corresponding to increasing values of ρ. The decisive quantity for the square-well potential is $\rho\sin\alpha$, which determines whether v is finite or vanishes. When $\rho \leq X$ we have $v = -v_0$ for all values of $\alpha \ \epsilon \ [0, \pi/2]$.

A. Short-distance behavior: $0 \leq \rho \leq R_0/\sqrt{2}$

The potential is now constant for all α and eq. (24) simplifies to

$$-\frac{\partial^2 \phi(\rho,\alpha)}{\partial \alpha^2} - (\rho^2 v_0 + \tilde{\lambda}(\rho))\phi(\rho,\alpha) = \frac{4}{\sqrt{3}}\rho^2 v_0 \int_{|\pi/3-\alpha|}^{\pi/2-|\pi/6-\alpha|} \phi(\rho,\alpha')d\alpha' \ . \tag{27}$$

The differential equation, without the right hand side, has vanishing solutions at $\alpha = 0$ of the form $\sin(k\alpha)$, which by insertion on the right hand side of eq. (27) easily is shown to remain of this form if k is an even integer, i.e. $k = 2n$, where n is an integer. A solution to eq. (27),

$$\phi(\rho, \alpha) \propto \sin(2n\alpha) \ , \tag{28}$$

is then by insertion found to correspond to the values of $\tilde{\lambda}_n$ given by

$$\tilde{\lambda}_n = 4n^2 - \rho^2 v_0 \left(1 - \frac{2}{n}(-)^n \frac{2}{\sqrt{3}} \sin(n\pi/3)\right) \ . \tag{29}$$

This wave function is furthermore also the solution for all potentials at $\rho = 0$ where the corresponding eigenvalue $\tilde{\lambda}_n = 4n^2$ then is obtained by demanding a vanishing wave function at $\alpha = \pi/2$. Since the potential only enters in combination with ρ^2, a perturbative solution to second order in ρ^2 is easily obtained for potentials, which are finite at the origin, by using $v(x) \approx v(0) = -v_0$. Thus eqs. (29) and (28) are also solutions for any potential in first order perturbation theory provided the depth of the square-well potential is replaced by the value $v(0)$ of the potential at $\rho = 0$.

The solution for $n = 2$ is independent of ρ and corresponds to the trivial solution (identically zero) to the Schrödinger equation. For $n > 2$ the solution is only an approximation, since the higher angular momenta (neglected here) contribute to these states.

B. Intermediate distances: $\pi/3 \leq \alpha_0 \leq \pi/2, R_0/\sqrt{2} \leq \rho \leq R_0\sqrt{2/3}$

The potential now vanishes for $\alpha_0 \leq \alpha \leq \pi/2$ (region II) where

$$\alpha_0 = \arcsin(R_0/\rho\sqrt{2}) \tag{30}$$

and it remains constant and finite for $0 \leq \alpha \leq \alpha_0$. The solution to eq. (24) in region II (vanishing potential) is therefore

$$\phi_{II} = A_{II} \sin\left((\alpha - \pi/2)\sqrt{\tilde{\lambda}}\right) = \frac{1}{2i} A_{II} \left(e^{i(\alpha-\pi/2)\sqrt{\tilde{\lambda}}} - e^{-i(\alpha-\pi/2)\sqrt{\tilde{\lambda}}}\right) \ , \tag{31}$$

where we already explicitly selected the solution vanishing at $\alpha = \pi/2$.

To proceed we divide the α-space into subregions:

$$\begin{aligned} &A: \ 0 \leq \alpha \leq \alpha_0 - \frac{\pi}{3} \\ &B: \ \alpha_0 - \frac{\pi}{3} \leq \alpha \leq \frac{2\pi}{3} - \alpha_0 \\ &C: \ \frac{2\pi}{3} - \alpha_0 \leq \alpha \leq \alpha_0 \\ &II: \alpha_0 \leq \alpha \leq \pi/2 \ . \end{aligned} \tag{32}$$

These regions are marked in fig. 1, which shows the integration limits of α' as function of α for the integral in eq. (24). The coupling between the different regions can then be seen and expressed in the set of equations:

$$-\frac{\partial^2 \phi_A(\rho,\alpha)}{\partial \alpha^2} - (\rho^2 v_0 + \tilde{\lambda}(\rho))\phi_A(\rho,\alpha) = \frac{4}{\sqrt{3}} \rho^2 v_0 \int_{|\pi/3-\alpha|}^{\pi/3+\alpha} \phi_C(\rho,\alpha')d\alpha' \ , \tag{33}$$

$$\begin{aligned} -\tfrac{\partial^2 \phi_C(\rho,\alpha)}{\partial \alpha^2} - (\rho^2 v_0 + \tilde{\lambda}(\rho))\phi_C(\rho,\alpha) = \tfrac{4}{\sqrt{3}} \rho^2 v_0 \Big(&\textstyle\int_{|\pi/3-\alpha|}^{\alpha_0-\pi/3} \phi_A(\rho,\alpha')d\alpha' \\ &+ \textstyle\int_{2\pi/3-\alpha_0}^{2\pi/3-\alpha} \phi_C(\rho,\alpha')d\alpha' + \int_{\alpha_0-\pi/3}^{2\pi/3-\alpha_0} \phi_B(\rho,\alpha')d\alpha'\Big) \ , \end{aligned} \tag{34}$$

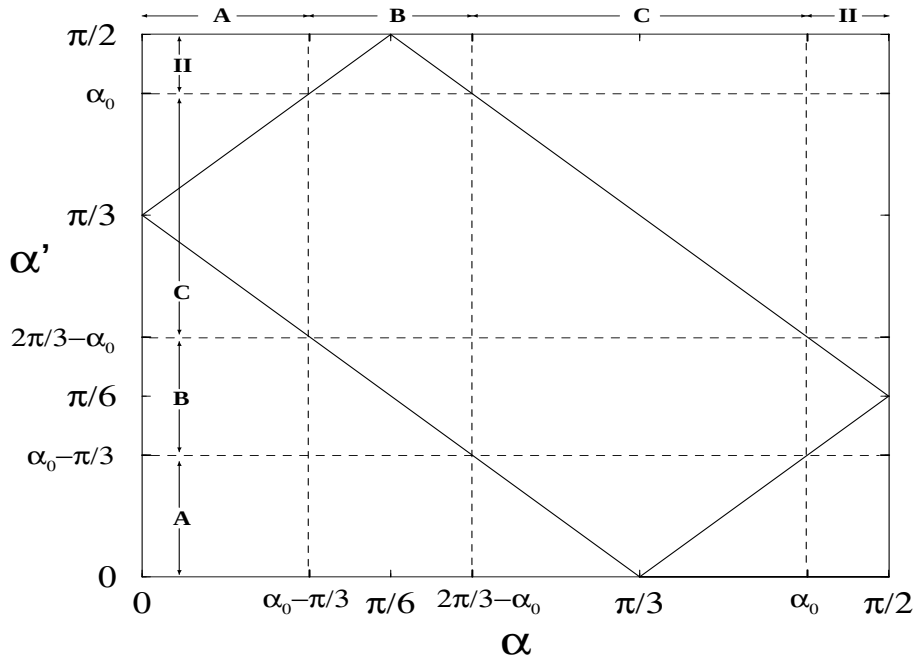

FIG. 1. The integration area in eq. (24) is the tilted rectangle (solid lines). The regions A,B,C and II (see eq. (32)) are marked on the figure. The size of α_0 corresponds to intermediate values of ρ: $R_0/\sqrt{2} \leq \rho \leq R_0\sqrt{2/3}$. The thick solid line on the axis indicates the range of α_0-values.

$$-\frac{\partial^2\phi_B(\rho,\alpha)}{\partial\alpha^2} - (\rho^2 v_0 + \tilde{\lambda}(\rho))\phi_B(\rho,\alpha) = \frac{4}{\sqrt{3}}\rho^2 v_0 \left(\int_{|\pi/3-\alpha|}^{2\pi/3-\alpha_0} \phi_B(\rho,\alpha')d\alpha' + \int_{\alpha_0}^{\pi/2-|\pi/6-\alpha|} \phi_{II}(\rho,\alpha')d\alpha' + \int_{2\pi/3-\alpha_0}^{\alpha_0} \phi_C(\rho,\alpha')d\alpha' \right) , \tag{35}$$

where the solutions are labeled according to subregion. Only ϕ_A and ϕ_C are directly coupled since ϕ_B enters in eq. (34) as an integral over a constant interval. Analogously only ϕ_B and ϕ_{II} are directly coupled.

Integrating the ϕ_{II}-term in eq. (35) we obtain exponentially increasing and decreasing functions, $\exp(\pm i\alpha\sqrt{\tilde{\lambda}})$, which by further integration and differential derivation still remain of the same functional form. Thus such functions matching ϕ_{II} are necessary in the solution, but in addition other exponentials are also possible. By insertion we then find that the wave function

$$\phi_B = B_+^{II} e^{i\alpha\sqrt{\tilde{\lambda}}} + B_-^{II} e^{-i\alpha\sqrt{\tilde{\lambda}}} + \sum_{k=1}^{3} (B_+^{(k)} e^{\alpha\kappa_B^{(k)}} + B_-^{(k)} e^{-\alpha\kappa_B^{(k)}}) \tag{36}$$

is a solution to eq. (35) when the B-coefficients and $\kappa_B^{(k)}$ are related as shown in appendix B. We have unique solutions for B_+^{II} and B_-^{II}, except the patological case of $\tilde{\lambda} = 16/3$, and three different solutions for $\kappa_B^{(k)}$ with three corresponding constraints between $B_+^{(k)}$ and $B_-^{(k)}$. This is the explanation for the three terms of the same form in eq. (36). In addition there is a link to the C-region providing one constraint between ϕ_C and the $B_\pm^{(k)}$-values.

In the coupled A- and C-regions we must also look for exponential functions as solutions. By insertion into eqs. (33) and (34) we find then that the wave functions

$$\phi_A = \sum_{k=0}^{3} A^{(k)} (e^{\alpha\kappa_{AC}^{(k)}} - e^{-\alpha\kappa_{AC}^{(k)}}) \tag{37}$$

$$\phi_C = \sum_{k=0}^{3} (C_+^{(k)} e^{\alpha\kappa_{AC}^{(k)}} + C_-^{(k)} e^{-\alpha\kappa_{AC}^{(k)}}) \tag{38}$$

indeed are solutions provided the A- and C-coefficients and the $\kappa_{AC}^{(k)}$-values are related as shown in appendix C. We have in eq. (37) already imposed the constraint that $\phi_A(\alpha = 0) = 0$ which eliminated one of the coefficients present in the other regions. We obtain four different solutions for $\kappa_{AC}^{(k)}$ as indicated by the four k-values and the two $C^{(k)}$-coefficients are uniquely determined by $A^{(k)}$. In addition there is a link to the B-region providing one constraint between ϕ_B and the $A^{(k)}$-values.

The solutions are now explicitly written down in the four regions named A,B,C and II. We have found wave functions containing the 8 parameters A_{II}, $B_-^{(k)}$, $A^{(0)}$ and $A^{(k)}$ for k=1,2,3. Two linear constraints exist between them as seen from eqs. (B11) and (B12) in appendix B and eqs. (C9) and (C10) in appendix C. The matching conditions at the three boundaries between the regions then provide additional 6 linear constraints on the remaining 6 free parameters. This leads as usual to the quantization condition for the eigenvalue $\tilde{\lambda}$.

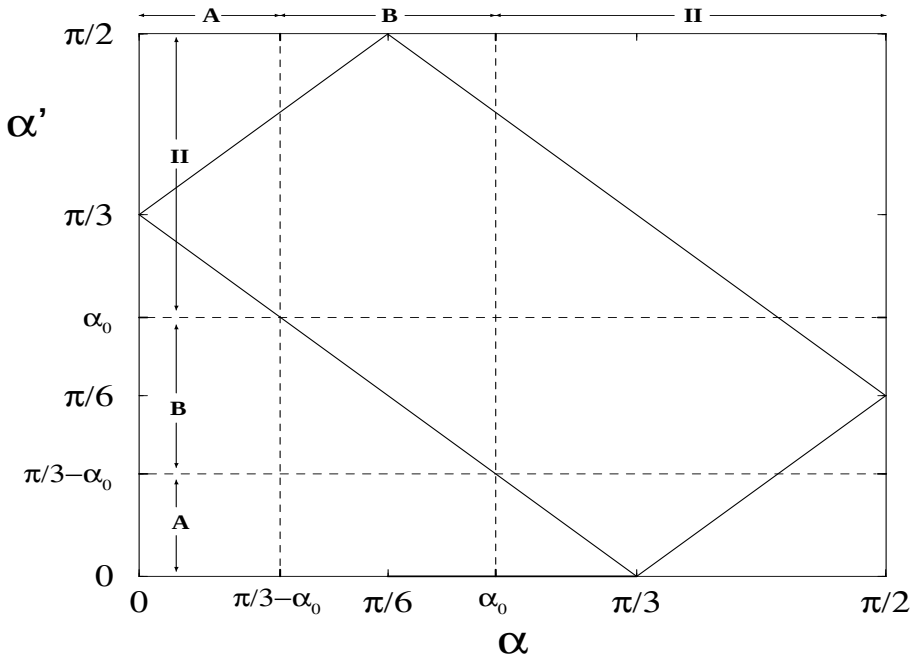

FIG. 2. The same as figure 1 for an α_0-value corresponding to intermediate values of ρ: $R_0\sqrt{2/3} \le \rho \le R_0\sqrt{2}$. The regions marked on the figure are defined in (see eq. (39)). The thick solid line on the axis indicates the range of α_0-values.

C. Intermediate distances: $\pi/6 \le \alpha_0 \le \pi/3\ ,\ R_0\sqrt{2/3} \le \rho \le R_0\sqrt{2}$

The potential again vanishes for $\alpha_0 \le \alpha \le \pi/2$, where α_0 is given in eq. (30). Subregion C has now been absorped into region II and subregions A and B are, as shown in fig. 2, now defined by

$$\begin{aligned} A &: 0 \le \alpha \le \frac{\pi}{3} - \alpha_0 \\ B &: \frac{\pi}{3} - \alpha_0 \le \alpha \le \alpha_0 \\ II &: \alpha_0 \le \alpha \le \pi/2\ . \end{aligned} \tag{39}$$

The coupled set of equations are now:

$$-\frac{\partial^2\phi_A(\rho,\alpha)}{\partial\alpha^2} - (\rho^2 v_0 + \tilde{\lambda}(\rho))\phi_A(\rho,\alpha) = \frac{4}{\sqrt{3}}\rho^2 v_0 \int_{\pi/3-\alpha}^{\pi/3+\alpha} \phi_{II}(\rho,\alpha')d\alpha'\ , \tag{40}$$

$$\begin{aligned} -\tfrac{\partial^2\phi_B(\rho,\alpha)}{\partial\alpha^2} - (\rho^2 v_0 + \tilde{\lambda}(\rho))\phi_B(\rho,\alpha) = \tfrac{4}{\sqrt{3}}\rho^2 v_0 \Big(&\int_{\pi/3-\alpha}^{\alpha_0} \phi_B(\rho,\alpha')d\alpha' \\ &+ \int_{\alpha_0}^{\pi/2-|\pi/6-\alpha|} \phi_{II}(\rho,\alpha')d\alpha'\Big)\ . \end{aligned} \tag{41}$$

The couplings are much simpler and essentially only ϕ_B enters in an integro-differential equation. Since ϕ_{II} still is given by the expression in eq. (31), a solution to eq. (40) would have to be proportional to $\sin(\alpha\sqrt{\tilde{\lambda}})$. The solution, where the right hand side vanishes, is analogously proportional to $\sin(\alpha\kappa)$, where

$$\kappa \equiv \sqrt{\rho^2 v_0 + \tilde{\lambda}(\rho)} \; . \tag{42}$$

By insertion we then obtain the complete solution to eq. (40) as

$$\phi_A = A_f(e^{i\alpha\sqrt{\tilde{\lambda}}} - e^{-i\alpha\sqrt{\tilde{\lambda}}}) + \frac{1}{2i}A_h(e^{i\alpha\kappa} - e^{-i\alpha\kappa}) = 2iA_f \sin\left(\alpha\sqrt{\tilde{\lambda}}\right) + A_h \sin(\alpha\kappa) \; , \tag{43}$$

where the coefficient A_h is arbitrary and A_f is given by

$$A_f = \frac{2A_{II}}{\sqrt{3\tilde{\lambda}}}(-e^{i\alpha\sqrt{\tilde{\lambda}}\,\pi/6} + e^{-i\alpha\sqrt{\tilde{\lambda}}\,\pi/6}) = -\frac{4iA_{II}}{\sqrt{3\tilde{\lambda}}} \sin\left(\alpha\sqrt{\tilde{\lambda}}\,\pi/6\right) \; . \tag{44}$$

The wave function in the B-region is a solution to eq. (41), which apart from a constant term arising from ϕ_C and the upper limit of the integral of ϕ_B, is identical to eq. (35). The solution is therefore given by eq. (36), which by insertion into eq. (41) also in this case leads to the expressions for the coefficients given in eqs. (B1)−(B4) in appendix B. The only difference is the constants in eq. (B5), which now is changed into

$$\begin{aligned} &\tfrac{A_{II}}{2\sqrt{\tilde{\lambda}}}\left(e^{i\sqrt{\tilde{\lambda}}(\alpha_0-\pi/2)} + e^{-i\sqrt{\tilde{\lambda}}(\alpha_0-\pi/2)}\right) + \tfrac{1}{i\sqrt{\tilde{\lambda}}}\left(B_+^{II}e^{i\sqrt{\tilde{\lambda}}\alpha_0} - B_-^{II}e^{-i\sqrt{\tilde{\lambda}}\alpha_0}\right) \\ &\qquad + \textstyle\sum_{k=1}^{3}\left[\tfrac{1}{\kappa_B^k}\left(B_+^{(k)}e^{\kappa_B^{(k)}\alpha_0} - B_-^{(k)}e^{-\kappa_B^{(k)}\alpha_0}\right)\right] = 0 \; . \end{aligned} \tag{45}$$

With the expressions in eqs. (B6), (B7) and (B10) for the coefficients we can rewrite the constraint from eq. (45) as

$$\begin{aligned} &\textstyle\sum_{k=1}^{3}\left[\frac{B_-^{(k)}}{\kappa_B^{(k)}}\left(\pm ie^{\kappa_B^{(k)}(\pi/3-\alpha_0)} - e^{-\kappa_B^{(k)}(2\pi/3-\alpha_0)}\right)\right] \\ &\qquad + \tfrac{A_{II}}{2\sqrt{\tilde{\lambda}}}\left[(e^{i\sqrt{\tilde{\lambda}}(\pi/2-\alpha_0)} + e^{-i\sqrt{\tilde{\lambda}}(\pi/2-\alpha_0)})\right. \\ &+\tfrac{1}{1-\frac{16}{3\tilde{\lambda}}}\;\tfrac{4}{i\sqrt{3\tilde{\lambda}}}\left.\left(e^{i\sqrt{\tilde{\lambda}}(\alpha_0-\pi/6)}\left(1-\tfrac{4}{i\sqrt{3\tilde{\lambda}}}\right) - e^{-i\sqrt{\tilde{\lambda}}(\alpha_0-\pi/6)}\left(1+\tfrac{4}{i\sqrt{3\tilde{\lambda}}}\right)\right)\right] = 0 \; . \end{aligned} \tag{46}$$

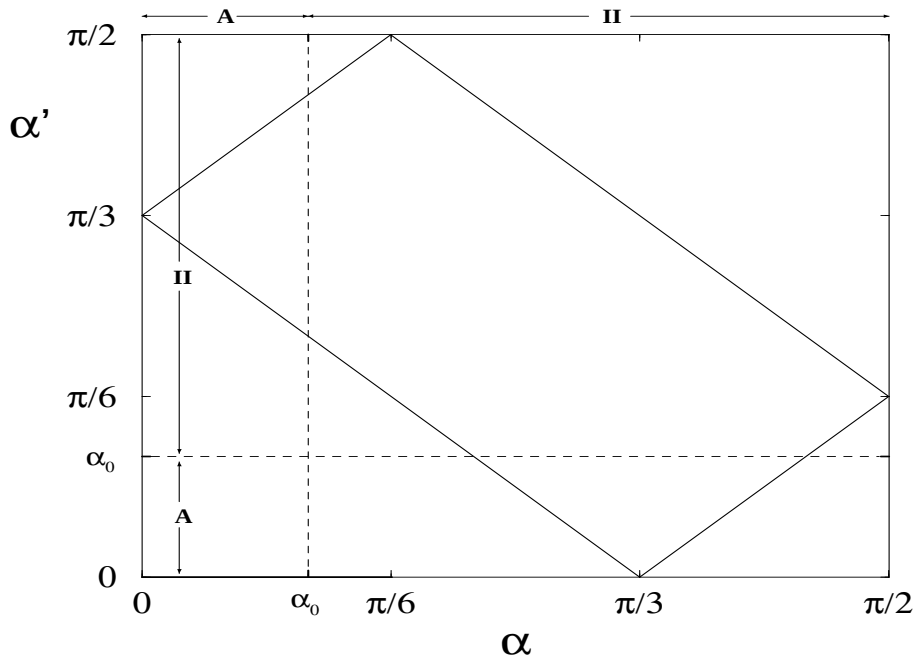

FIG. 3. The same as figure 1 for an α_0-value corresponding to large values of ρ: $R_0\sqrt{2} \leq \rho \leq \infty$. The regions marked on the figure are defined in (see eq. (47)). The thick solid line on the axis indicates the range of α_0-values.

We are now left with 5 parameters, i.e. A_{II}, A_h and $B_-^{(k)}$ for k=1,2,3. One of these is determined by the constraint in eq. (46). The matching conditions at the two boundaries between the regions provide additional 4 linear constraints on the remaining 4 free parameters. Therefore we obtain again a quantization condition for the eigenvalue $\tilde{\lambda}$.

D. Large-distance behavior: $0 \le \alpha_0 \le \pi/6$, $R_0\sqrt{2} \le \rho \le \infty$

The potential vanishes as usual for $\alpha_0 \le \alpha \le \pi/2$, where α_0 again is given in eq. (30). Subregion B has now also been absorped into region II and subregion A is, as shown in fig. 3, now defined by

$$\begin{aligned} &A : 0 \le \alpha \le \alpha_0 \\ &II : \alpha_0 \le \alpha \le \pi/2 \; . \end{aligned} \tag{47}$$

The equations now reduce to eq. (40), which couples ϕ_A and ϕ_{II}. The latter is again given by the expression in eq. (31) and consequently ϕ_A is given by eq. (43) with one arbitrary coefficient and the other coefficient expressed in eq. (44).

The matching conditions at α_0 provide 2 linear constraints on the remaining 2 free parameter and we obtain again a quantization condition for the eigenvalue $\tilde{\lambda}$. An explicit expression is most conveniently derived by rewriting the solution in eq. (43) as

$$\phi_A = \frac{8A_{II}}{\sqrt{3\tilde{\lambda}}} \sin(\sqrt{\tilde{\lambda}}\,\pi/6) \sin(\alpha\sqrt{\tilde{\lambda}}) + A_h \sin(\alpha\kappa) \; . \tag{48}$$

Matching the wave functions in eqs. (31) and (48) and their first derivatives at $\alpha = \alpha_0$ gives immediately

$$\frac{8A_{II}}{\sqrt{3\tilde{\lambda}}} \sin(\sqrt{\tilde{\lambda}}\,\pi/6) \sin(\alpha_0\sqrt{\tilde{\lambda}}) + A_h \sin(\alpha_0\kappa) = A_{II} \sin\left((\alpha_0 - \pi/2)\sqrt{\tilde{\lambda}}\right) \tag{49}$$

$$\frac{8A_{II}}{\sqrt{3}} \sin(\sqrt{\tilde{\lambda}}\,\pi/6) \cos(\alpha_0\sqrt{\tilde{\lambda}}) + A_h\kappa \cos(\alpha_0\kappa) = A_{II}\sqrt{\tilde{\lambda}} \cos\left((\alpha_0 - \pi/2)\sqrt{\tilde{\lambda}}\right) \; . \tag{50}$$

The determinant D corresponding to this linear set of equations for A_{II} and A_h is

$$\begin{aligned} D = &\tfrac{\kappa}{\sqrt{\tilde{\lambda}}} \cos(\alpha_0\kappa) \left[\tfrac{8}{\sqrt{3}} \sin(\sqrt{\tilde{\lambda}}\,\pi/6) \sin(\alpha_0\sqrt{\tilde{\lambda}}) - \sqrt{\tilde{\lambda}} \sin\left((\alpha_0 - \pi/2)\sqrt{\tilde{\lambda}}\right)\right] \\ &- \sin(\alpha_0\kappa) \left[\tfrac{8}{\sqrt{3}} \sin(\sqrt{\tilde{\lambda}}\,\pi/6) \cos(\alpha_0\sqrt{\tilde{\lambda}}) - \sqrt{\tilde{\lambda}} \cos\left((\alpha_0 - \pi/2)\sqrt{\tilde{\lambda}}\right)\right] \; . \end{aligned} \tag{51}$$

The non-trivial solutions are obtained for $D = 0$, which therefore is the eigenvalue equation for $\tilde{\lambda}$.

We can simplify even further in the limit of very large distances, where

$$\alpha_0 \approx \frac{R_0}{\rho\sqrt{2}} \; , \quad \kappa \approx \rho\sqrt{v_0} \; , \quad \kappa\alpha_0 \approx R_0\sqrt{v_0/2} \; , \tag{52}$$

where we assumed that $\tilde{\lambda}$ remains finite when ρ increases towards infinity. The eigenvalue equation $D = 0$ obtained from eq. (51) by expansion to the lowest orders in $1/\rho$ then becomes

$$\frac{8}{\sqrt{3}} \sin(\sqrt{\tilde{\lambda}}\,\pi/6) - \sqrt{\tilde{\lambda}} \cos(\sqrt{\tilde{\lambda}}\,\pi/2) = \frac{\rho\sqrt{2}}{a_{scat}} \sin(\sqrt{\tilde{\lambda}}\,\pi/2) \; , \tag{53}$$

where we introduced the two-body scattering length a_{scat} of the square-well potential

$$\frac{1}{a_{scat}} = \frac{\sqrt{v_0/2}}{-R_0\sqrt{v_0/2} + \tan(R_0\sqrt{v_0/2})} \; . \tag{54}$$

The factor of two appearing here is due to the mass (not the reduced mass) in the definition of v_0. When ρ approaches infinity eq. (53) can be valid only if $\tilde{\lambda} \to 4n^2$, where n must be an integer. This is the same spectrum as obtained in eq. (29) for $\rho = 0$. Expanding to the next order in $1/\rho$ gives

$$\tilde{\lambda}_n = 4n^2\left(1 - \frac{12a_{scat}}{\pi\rho\sqrt{2}}\right) . \tag{55}$$

If the scattering length is infinitly large eq. (53) reduces to the equation derived by Efimov [4], which instead has the solution $\tilde{\lambda} = -1.01251$ leading to infinitly many bound states obtained from eq. (7). Clearly if need be these expressions can be extended to any order of correction in $1/\rho$.

When $\tilde{\lambda}/\rho^2$ diverges the asymptotic behavior is different. We then define ϵ_x in the large-distance limit by $\tilde{\lambda} \to 2m\epsilon_x\rho^2/\hbar^2 \equiv \bar{\epsilon_x}\rho^2$. It is easy to see that eq. (51) at most only can have solutions for negative ϵ_x. Instead of eq. (52) we then at large ρ obtain

$$\begin{aligned}\sqrt{\tilde{\lambda}} \approx -i\rho\sqrt{-\bar{\epsilon_x}}\ , \quad \kappa \approx \rho\sqrt{v_0+\bar{\epsilon_x}}\ , \quad \alpha_0\sqrt{\tilde{\lambda}} \approx -iR_0\sqrt{-\bar{\epsilon_x}/2}\ , \\ \frac{\kappa}{\sqrt{\tilde{\lambda}}} \approx i\sqrt{-(v_0+\bar{\epsilon_x})/\bar{\epsilon_x}}\ , \quad \kappa\alpha_0 \approx R_0\sqrt{(v_0+\bar{\epsilon_x})/2}\,.\end{aligned} \tag{56}$$

Expanding eq. (51) to lowest order in $1/\rho$ we find (instead of eq. (53))

$$\tan\left(R_0\sqrt{(v_0+\bar{\epsilon_x})/2}\right) \approx -\sqrt{\frac{v_0+\bar{\epsilon_x}}{-\bar{\epsilon_x}}}\,, \tag{57}$$

which is the eigenvalue equation for the two-body subsystem with an energy corresponding to $\bar{\epsilon_x}$. If two-body bound states exist ϵ_x must be one of the energies. In other words for each bound two-body state one $\tilde{\lambda}-$value diverges parabolically according to eq. (56).

For general short-range potentials, which vanish at distances larger than a given finite radius R_0, we obtain again the solutions described in this subsection. The Efimov conditions in particular therefore remain the same for such potentials.

E. Large-distance behavior of $P_{nn'}$ and $Q_{nn'}$

According to eqs. (8) and (9) we must find the derivative of the Schrödinger wave function, $\Phi_n \equiv \sum_{i=1}^{3}\phi_n^{(i)}$, with respect to ρ. In the present symmetric case the three Faddeev components are equal. We choose to work in the first Jacobi coordinate system and must then express the orther two components in this set of coordinates. The transformation is given in eq. (11) and the angular wave function is

$$\Phi \propto \phi_n(\alpha_1) + 2R_{12}\left[\frac{\phi_n}{\sin(2\alpha_2)}\right] = \phi_n(\alpha_1) + \frac{4}{\sqrt{3}}\int_{|\pi/3-\alpha_1|}^{\pi/2-|\pi/6-\alpha_1|}\phi_n(\rho,\alpha_2)d\alpha_2\ , \tag{58}$$

where the factor 2 arises since the R_{13}-operation gives the same result as that of R_{12}. The wave function $\phi_n(\alpha_1)$ explicitly is shown in eqs. (31) and (48) with A_h obtained from eq. (49) as

$$N(\sqrt{\tilde{\lambda}},\kappa,\alpha_0) \equiv \frac{A_h}{A_{II}} = \frac{1}{\sin(\alpha_0\kappa)}\left[\frac{8}{\sqrt{3\tilde{\lambda}}}\sin(\sqrt{\tilde{\lambda}}\,\pi/6)\sin(\alpha_0\sqrt{\tilde{\lambda}}) - \sin\left((\alpha_0-\pi/2)\sqrt{\tilde{\lambda}}\right)\right]\ . \tag{59}$$

It is then clear that Φ_n only depends on ρ through $\sqrt{\tilde{\lambda}_n}, \kappa_n$ and α_0. Thus

$$\frac{\partial}{\partial\rho}\Phi_n = \frac{\partial\sqrt{\tilde{\lambda}_n}}{\partial\rho}\frac{\partial\Phi_n}{\partial\sqrt{\tilde{\lambda}_n}} + \frac{\partial\kappa_n}{\partial\rho}\frac{\partial\Phi_n}{\partial\kappa_n} + \frac{\partial\alpha_0}{\partial\rho}\frac{\partial\Phi_n}{\partial\alpha_0}\ . \tag{60}$$

For $P_{nn'}$ we immediately obtain that

$$P_{nn'} \propto \frac{\partial\alpha_0}{\partial\rho}\int_0^{\pi/2} d\alpha_1\Phi_n^*\frac{\partial}{\partial\alpha_0}\Phi_{n'} \tag{61}$$

as seen from the orthogonality of the different Φ_n. Furthermore, the derivative of the overall normalization factor A_{II}, which also depends on α_0, does not contribute to eq. (61) again due to the orthogonality of the wave functions. The large-distance behavior of $\partial\alpha_0/\partial\rho$ is seeen from the definition in eq. (30) to be $\propto 1/\rho^2$.

We now divide the integration interval into the four parts appropriate for the combination of the transformation between Jacobi coordinates contained in eq. (58) and the different wave functions given in the two intervals of eq. (47), i.e.

$$I_1 = [0, \alpha_0] \ , \ I_2 = [\alpha_0, \pi/3 - \alpha_0] \ , \ I_3 = [\pi/3 - \alpha_0, \pi/3 + \alpha_0] \ , \ I_4 = [\pi/3 + \alpha_0, \pi/2] \ . \tag{62}$$

As seen from eqs. (31) and (48) and fig.3, the wave function in eq. (61) is apart from A_{II} independent of α_0 in the intervals I_2 and I_4, where the contribution to $P_{nn'}$ therefore is zero.

The third interval vanishes with $\alpha_0 \propto 1/\rho$ and the α_0-dependence of the wave function (again apart from A_{II}) arises entirely from the A_h-term of the two transformed Faddeev components. The derivative is proportional to $\kappa\alpha_0^2 \partial N/\partial\alpha_0$, which vanishes as $\alpha_0 \propto 1/\rho$. The contribution to $P_{nn'}$ from the third interval is then vanishing at least as fast as $1/\rho^4$.

The size of the first interval I_1 vanishes as $\alpha_0 \propto 1/\rho$. The α_0-dependence of the wave function arises (apart from A_{II}) entirely from the A_h-term in the first Faddeev component. The derivative of N is proportional to $\kappa\alpha_0$, which remains finite together with the total wave function. Therefore $P_{nn'}$ in eq. (61) vanishes as $1/\rho^3$ for all n, n'. The leading order term comes entirely from I_1, where the transformed Faddeev components combined with the derivative of the A_h-term of the first component contribute to the order $1/\rho^3$.

For the non-diagonal $Q_{nn'}$ we also in analogy with eq. (61) obtain

$$Q_{nn'} \propto \left(\frac{\partial\alpha_0}{\partial\rho}\right)^2 \int_0^{\pi/2} d\alpha_1 \Phi_n^* \frac{\partial^2}{\partial\alpha_0^2} \Phi_{n'} + \frac{\partial^2\alpha_0}{\partial\rho^2} \int_0^{\pi/2} d\alpha_1 \Phi_n^* \frac{\partial}{\partial\alpha_0} \Phi_{n'} \ , \tag{63}$$

where both terms therefore approach zero at least as $1/\rho^4$. The first term due to $(\partial\alpha_0/\partial\rho)^2 \propto 1/\rho^4$ and the last term due to $\partial^2\alpha_0/\partial\rho^2 \propto 1/\rho^3$ and the integral found in eq. (61) vanishing as $1/\rho$.

The diagonal term Q_{nn} can be written as a sum of six terms corresonding to eq. (63) and the analogous terms where α_0 is replaced by $\sqrt{\tilde{\lambda}_n}$ and κ_n. The three terms with first order derivatives of the wave function all vanish due to the normalization of Φ_n. Since $(\partial\alpha_0/\partial\rho)^2 \propto 1/\rho^4$, we are left with the leading terms

$$Q_{nn} \propto \left(\frac{\partial\kappa_n}{\partial\rho}\right)^2 \int_0^{\pi/2} d\alpha_1 \Phi_n^* \frac{\partial^2}{\partial\kappa_n^2} \Phi_n + \left(\frac{\partial\sqrt{\tilde{\lambda}_n}}{\partial\rho}\right)^2 \int_0^{\pi/2} d\alpha_1 \Phi_n^* \frac{\partial^2}{\partial(\sqrt{\tilde{\lambda}_n})^2} \Phi_n \ , \tag{64}$$

which for $n \neq n'$ would vanish due to orthogonality. The factor $(\partial\kappa_n/\partial\rho)^2$ approaches a constant when $\rho \to \infty$ and so does $(\partial\sqrt{\tilde{\lambda}_n}/\partial\rho)^2$ when $\sqrt{\tilde{\lambda}_n}$ corresponds to a bound two-body state and otherwise $(\partial\sqrt{\tilde{\lambda}_n}/\partial\rho)^2 \propto 1/\rho^4$.

We again divide into the appropriate four intervals I_1, I_2, I_3, I_4. Let us first study the case without two-body bound states where the leading order term arises from the κ-derivation, since then $(\partial\sqrt{\tilde{\lambda}_n}/\partial\rho)^2 \propto 1/\rho^4$. The second and fourth intervals contain no κ-dependence and the contribution consequently vanish. The third interval contributes at the most to the order $1/\rho^4$, i.e. $\alpha_0 \propto 1/\rho$ from the interval size, $\alpha_0^2 \propto 1/\rho^2$ from the second derivative of the A_h-term of the two transformed Faddeev components and finally α_0 from the interval size of the transformation integral. In the first interval we have $\alpha_0 \propto 1/\rho$ from the size of the interval and $\alpha_0^2 \propto 1/\rho^2$ from the second derivative of the A_h-term in the first Faddeev component. The leading term of the total wave function arises from the two tranformed Faddeev components and it approaches a finite constant for $\rho \to \infty$.

The κ_n-dependence of the normalization constant A_{II} did not contribute to $P_{nn'}$ due to orthogonality of the wave functions. However, for Q_{nn} we must consider this dependence. If we define $\Phi_n \equiv A_{II}\tilde{\Phi}_n$, it is easy to show that

$$\frac{\partial A_{II}}{\partial\kappa_n} = -|A_{II}|^3 \int_0^{\pi/2} d\alpha_1 \tilde{\Phi}_n^* \frac{\partial}{\partial\kappa_n} \tilde{\Phi}_n \ . \tag{65}$$

Going through the different intervals we find again vanishing contributions from I_2 and I_4, a contribution proportional to $1/\rho^3$ from I_3 and the leading order term $\propto 1/\rho^2$ from I_1 arising from the two transformed Faddeev components combined with the derivative of the first Faddeev component. We therefore have $\partial A_{II}/\partial\kappa_n \propto 1/\rho^2$. Furthermore, since

$$\begin{aligned} \int_0^{\pi/2} d\alpha_1 \Phi_n^* \frac{\partial^2}{\partial\kappa_n^2} \Phi_n = 2A_{II} \frac{\partial A_{II}}{\partial\kappa_n} \int_0^{\pi/2} d\alpha_1 \tilde{\Phi}_n^* \frac{\partial}{\partial\kappa_n} \tilde{\Phi}_n \\ + A_{II} \frac{\partial^2 A_{II}}{\partial\kappa_n^2} \int_0^{\pi/2} d\alpha_1 \tilde{\Phi}_n^* \tilde{\Phi}_n + |A_{II}|^2 \int_0^{\pi/2} d\alpha_1 \tilde{\Phi}_n^* \frac{\partial^2}{\partial\kappa_n^2} \tilde{\Phi}_n \end{aligned} \tag{66}$$

the terms with derivatives of A_{II} approach zero at least as fast as $1/\rho^3$ and so did the last term as discussed above. Thus Q_{nn} approaches zero at least as fast as $1/\rho^3$.

In the case where $\tilde{\lambda}_n$ corresponds to a bound two-body state, we get the same results for the same reasons as described above. However, then $(\partial\sqrt{\tilde{\lambda}_n}/\partial\rho)^2$ approaches a constant for $\rho \to \infty$ and the second term in eq. (64) must also be considered. Although possible this is a rather elaborate procedure due to the many terms containing $\tilde{\lambda}_n$. We shall instead give arguments based directly on the equation of motion.

At large distances the Faddeev components decouple as for example seen from eq. (24) where the potential in the last term vanishes unless α approaches zero at least as fast as $1/\rho$. However, when $\alpha \to 0$ the size of the integration interval vanishes together with the integral itself. Thus the angular wave function is in this limit determined by an ordinary Schrödinger equation, i.e.

$$\left[-\frac{d^2}{dz^2} - \frac{\tilde{\lambda}}{\rho^2} + v(z)\right]\phi(z) = 0 \tag{67}$$

where we introduced the new variable $z = \rho\sin\alpha \approx \rho\alpha$. The boundary conditions $\phi(z =\!\!\rightarrow 0) = \phi(z \to \infty) = 0$ make eq. (67) equivalent to the Schrödinger equation for one of the (three identical) two-body subsystems, i.e.

$$\left[-\frac{d^2}{dz^2} + v(z)\right]u(z) = \frac{2m\epsilon_x}{\hbar^2}u(z) \tag{68}$$

where $z = r\sqrt{m/\mu}$, r is the relative coordinate, ϵ_x is the energy, μ the reduced mass of the two-body system.

From eqs. (67) and (68) we obtain

$$\tilde{\lambda} = \lambda + 4 = \frac{2m\epsilon_x}{\hbar^2}\rho^2 + \int dz u(z)\left[z\frac{du}{dz} + z^2\frac{d^2u}{dz^2}\right] \tag{69}$$

and $\phi(z) = \sqrt{\rho}u(z)$. Introducing $\phi(z)$ in the Q_{nn}-equation we get

$$Q_{nn} = -\frac{1}{4\rho^2} + \frac{1}{\rho^2}\int dz u(z)\left[z\frac{du}{dz} + z^2\frac{d^2u}{dz^2}\right] . \tag{70}$$

The integrals in eqs. (69) and (70) cancel in the combination $(\lambda + 15/4)/\rho^2 - Q_{nn}$ found in the coupled set of radial equations, see eq. (7). The result, $2m\epsilon_x/\hbar^2$, restores the proper two-body radial asymptotics which describes the motion of one of the particles against the bound system of the other two particles.

We can conclude that the coupled set of radial equations in eq. (7) decouple in the limit of large ρ, since both coupling terms $P_{nn'}$ and $Q_{nn'}$ approach zero at least as fast as $1/\rho^3$, i.e. faster than that of the leading centrifugal barrier term $1/\rho^2$. This behavior at large distances is a general result valid for all short-range potentials.

IV. THREE NON-IDENTICAL PARTICLES FOR SPIN INDEPENDENT INTERACTIONS

In the asymmetric case with spin independent interactions the three Faddeev equations in eq. (16) can by use of eqs. (11), (12), (17) and (19) explicitly be written as

$$\begin{aligned} &-\tfrac{\partial^2\phi^{(i)}(\rho,\alpha_i)}{\partial\alpha_i^2} + (\rho^2 v_i(\rho\sin\alpha_i) - \tilde{\lambda}(\rho))\phi^{(i)}(\rho,\alpha_i) + \rho^2 v_i(\rho\sin\alpha_i)\left(\tfrac{1}{\sin(2\varphi_j)}\right. \\ &\left.\times\int_{|\varphi_j-\alpha_i|}^{\pi/2-|\pi/2-\varphi_j-\alpha_i|}\phi^{(k)}(\rho,\alpha_k)d\alpha_k + \tfrac{1}{\sin(2\varphi_k)}\int_{|\varphi_k-\alpha_i|}^{\pi/2-|\pi/2-\varphi_k-\alpha_i|}\phi^{(j)}(\rho,\alpha_j)d\alpha_j\right) = 0 , \end{aligned} \tag{71}$$

where $i = 1, 2, 3$ and where we again omitted the label "n". We shall follow the same procedure as in the symmetric case and mostly use square-well potentials, i.e.

$$V_i(r) = -V_0^{(i)}\Theta(r < R_i) \tag{72}$$

and the reduced potentials in eq. (71) are therefore the other stepfunctions

$$v_i(x) = -v_0^{(i)}\Theta(x < X_i) , \tag{73}$$

where $v_0^{(i)} = 2mV_0^{(i)}/\hbar^2$ and $X_i = R_i\mu_{jk}$. The angular Faddeev equation in eq. (71) are now solved analytically in different intervals corresponding to increasing values of ρ. The decisive quantity for each of the square-well potentials is $\rho\sin\alpha_i$, which determines whether $v_i(x)$ is finite or vanishes. When $\rho \leq X_i$ we have $v_i(x) = -v_0^{(i)}$ for all values of $\alpha_i \,\epsilon\, [0, \pi/2]$.

Without loss of generality we can assume that $X_1 \leq X_2 \leq X_3$ which implies that $\alpha_0^{(1)} \leq \alpha_0^{(2)} \leq \alpha_0^{(3)}$ where in analogy to eq. (30) we define

$$\alpha_0^{(i)} \equiv \arcsin(X_i/\rho)\ . \tag{74}$$

We shall in this section confine ourselves to small or large distances. The intermediate distances can be solved in analogy to the symmetric case by division into ρ-intervals. The related eigenfunctions are combinations of simple functions, but they are rather tedious to write down in their full length and less interesting as well.

A. Short-distance behavior: $0 \leq \rho \leq X_1$

The potentials are now all constants for all α-values and eq. (71) is an inhomogeneous differential equation for $\phi^{(i)}$, where the inhomogeneous part contains $\phi^{(j)}$ and $\phi^{(k)}$. It can therefore be solved by adding one inhomogeneous solution

$$\phi^{(i)}(\rho, \alpha_i) = a_i \sin(2n\alpha_i) \tag{75}$$

to the complete set of homogeneous solutions

$$\phi^{(i)}(\rho, \alpha_i) = b_i \sin(\alpha_i\kappa_i)\ \ ,\ \kappa_i = \sqrt{v_0^{(i)}\rho^2 + \tilde{\lambda}(\rho)}\ , \tag{76}$$

where we already imposed the boundary condition $\phi^{(i)}(\rho, \alpha_i = 0) = 0$.

The inhomogeneous components in eq. (75) must all be proportional to $\sin(2n\alpha)$, since the integrated values in eq. (71) otherwise do not return the same function as required by this equation. The homogeneous components in eq. (76) can only be solutions if $v_0^{(1)} = v_0^{(2)} = v_0^{(3)}$ and $\varphi_1 = \varphi_2 = \varphi_3 = \pi/3$ which is the symmetric case discussed in section 3. Any asymmetry therefore leads to $b_1 = b_2 = b_3 = 0$. A set of solutions is therefore obtained, if and only if the three equations

$$\rho^2(v_0^{(i)} - \epsilon)a_i = \rho^2 v_0^{(i)}(a_k d_j + a_j d_k) \tag{77}$$

are fulfilled when $\{i, j, k\}$ are the three permutations of $\{1, 2, 3\}$ and where we have defined

$$\tilde{\lambda} \equiv 4n^2 - \rho^2\epsilon \quad , \quad d_i \equiv -\frac{\sin(2n\varphi_i)}{n\sin(2\varphi_i)}\ . \tag{78}$$

In the limiting case of $\rho = 0$ we immediately find the three times degenerate solution $\tilde{\lambda} = 4n^2$.

Non-trivial solutions for $\rho \neq 0$ only occur when the corresponding determinant vanishes, i.e.

$$\begin{vmatrix} \epsilon - v_0^{(1)} & v_0^{(1)}d_3 & v_0^{(1)}d_2 \\ v_0^{(2)}d_3 & \epsilon - v_0^{(2)} & v_0^{(2)}d_1 \\ v_0^{(3)}d_2 & v_0^{(3)}d_1 & \epsilon - v_0^{(3)} \end{vmatrix} = 0\ , \tag{79}$$

which determines the possible $\tilde{\lambda}(\rho)$.

In the symmetric case ($v_0^{(1)} = v_0^{(2)} = v_0^{(3)} \equiv v_0$, $d_1 = d_2 = d_3 \equiv d_0$) we obtain the two solutions

$$\tilde{\lambda}(\rho) = 4n^2 - \rho^2 v_0(1 + d_0) \tag{80}$$

$$\tilde{\lambda}(\rho) = 4n^2 - \rho^2 v_0(1 - 2d_0) \tag{81}$$

where the first is two times degenerate and the last corresponds to the symmetric solution as seen from eq. (77).

In the general case we define

$$S_0 \equiv v_0^{(1)} v_0^{(2)} v_0^{(3)} (2d_1 d_2 d_3 + d_1^2 + d_2^2 + d_3^2 - 1)$$
$$S_1 \equiv v_0^{(1)} v_0^{(2)} (1 - d_3^2) + v_0^{(1)} v_0^{(3)} (1 - d_2^2) + v_0^{(2)} v_0^{(3)} (1 - d_1^2) \tag{82}$$
$$S_2 \equiv -v_0^{(1)} - v_0^{(2)} - v_0^{(3)}$$

and rewrite eq. (79) as

$$\epsilon^3 + S_2 \epsilon^2 + S_1 \epsilon + S_0 = 0 \ , \tag{83}$$

which is independent of ρ. Thus the short-distance behavior of $\tilde{\lambda}(\rho)$ is quadratic in ρ and given by eq. (78).

For $n = 1$ we have $d_i = -1$, $S_0 = S_1 = 0$ and consequently the solutions $\epsilon = -S_2$ and the doubly degenerate spurious solution $\epsilon = 0$. This is also seen from eq. (80) with $n = 1$. The corresponding wave function is determined by eq. (75) with $a_i \propto v_0^{(i)}$ as seen from eq. (77).

For $n = 2$ we have

$$d_i = -\cos(2\varphi_i) = \frac{\tan^2 \varphi_i - 1}{\tan^2 \varphi_i + 1} \tag{84}$$

and $S_0 = 0$. Consequently we find the spurious solution $\epsilon = 0$ and the two solutions

$$\epsilon = \frac{1}{2} \left(-S_2 \pm \sqrt{S_2^2 - 4S_1} \right) \ , \tag{85}$$

where ϵ is real, since $S_2^2 \geq 4S_1$ for all masses and all values of the strength parameters $v_0^{(i)}$. The related wave functions are found from eq. (77).

For arbitrary n-values exceeding 2 we have three real solutions to eq. (83) as discussed in appendix D. For $n \to \infty$, we have $d_i \to 0$.

Since the potentials also as in the symmetric case only enter in combination with ρ^2, a perturbative solution to first order in ρ^2 is obtained for arbitrary potentials by using $v_i(x) \approx v_i(0) = -v_0^{(i)}$. Thus the solutions in this subsection are also solutions for any potential in first order perturbation theory provided the depth of the square-well potential is replaced by the value $v_i(0)$ of the potential at $\rho = 0$.

B. Large-distance behavior

We define in this connection large distances to mean that

$$\alpha_0^{(1)} \leq \alpha_0^{(2)} \leq \alpha_0^{(3)} \leq |\varphi_1 - \alpha_0^{(1)}| \leq |\varphi_2 - \alpha_0^{(2)}| \leq |\varphi_3 - \alpha_0^{(3)}| \tag{86}$$

for all sets $1, 2, 3$, which implies that $\alpha_0^{(i)} \leq \frac{1}{2}\mathrm{Min}(\varphi_i)$ for all sets $1, 2, 3$. The potentials all vanish for $\alpha_i \geq \alpha_0^{(i)}$ and the solutions to eq. (71) are therefore the same as in eq. (31), i.e.

$$\phi^{(i)}(\alpha_i) = a_i \sin\left((\alpha_i - \pi/2)\sqrt{\tilde{\lambda}} \right) \ . \tag{87}$$

In the other region where $\alpha_i \leq \alpha_0^{(i)}$ we also obtain decoupled solutions, i.e.

$$\phi^{(i)}(\alpha_i) = b_i \sin(\alpha_i \kappa_i) + c_i \sin\left(\alpha_i \sqrt{\tilde{\lambda}} \right) \ , \tag{88}$$

where the first term is the solution to the homogeneous equation. The constants b_i are therefore completely free, since the functions to be integrated in eq. (71) only involve the functions in eq. (87), i.e. values of $\alpha_i \geq \alpha_0^{(i)}$, i=1,2,3.

The connection between a_i and c_i are now found by insertion of eqs. (87) and (88) into eq. (71). The result is

$$\bar{c}_i = \frac{2F^3}{\sqrt{\tilde{\lambda}}} (\bar{a}_j + \bar{a}_k) \ , \tag{89}$$

where

$$\bar{a}_i = a_i/f_i \ , \ \bar{c}_i = -c_i f_i \ , \ f_i = \frac{\sin\left((\varphi_i - \pi/2)\sqrt{\tilde{\lambda}}\right)}{\sin(2\varphi_i)} \ , \ F = (f_1 f_2 f_3)^{1/3} \ . \tag{90}$$

This determines c_i from given values of a_i.

The matching conditions at $\alpha_0^{(1)}, \alpha_0^{(2)}$ and $\alpha_0^{(3)}$ provide 6 linear constraints between the parameters a_i and b_i, where c_i is found from eqs. (89)-(90). They only have non-trivial solutions when the corresponding determinant vanishes. In appendix E is shown that this condition, which determines the eigenvalues $\tilde{\lambda}$, can be formulated as

$$D = B_1 B_2 B_3 + 2A_1 A_2 A_3 - B_1 A_2 A_3 - A_1 A_2 B_3 - A_1 B_2 A_3 = 0 \ , \tag{91}$$

where A_i and B_i are defined in appendix E.

In the symmetric case, ($B_1 = B_2 = B_3 = B_0$, $A_1 = A_2 = A_3 = A_0, f_1 = f_2 = f_3$), we have $D = (A_0 - B_0)^2(2A_0 + B_0)$, where the solutions to $D = 0$ correspond to those of eqs. (80) and (81).

Eq. (91) simplifies in the limit of very large distances, where

$$\alpha_0^{(i)} \approx \frac{X_i}{\rho} \ , \ \ \kappa_i \approx \rho\sqrt{v_0^{(i)}} \ , \ \ \kappa_i \alpha_0^{(i)} \approx X_i \sqrt{v_0^{(i)}} \ , \tag{92}$$

where we assumed that $\tilde{\lambda}$ remains finite. Expansion to lowest order in $1/\rho$ then gives, see eqs. (E4) and (E7),

$$A_i \approx -2F\mu_{jk}\sqrt{v_0^{(i)}} a_{scat}^{(i)} \cos(X_i\sqrt{v_0^{(i)}}) \ , \tag{93}$$

$$B_i \approx -\frac{f_i^2}{F^2}\cos(X_i\sqrt{v_0^{(i)}})\sqrt{v_0^{(i)}}\left(\rho\sin(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) + a_{scat}^{(i)}\mu_{jk}\sqrt{\tilde{\lambda}}\cos(\frac{\pi}{2}\sqrt{\tilde{\lambda}})\right) \ , \tag{94}$$

where the scattering length for the two-body system in analogy with eq. (54) is given by

$$\frac{1}{a_{scat}^{(i)}} = \frac{\mu_{jk}\sqrt{v_0^{(i)}}}{-X_i\sqrt{v_0^{(i)}} + \tan\left(X_i\sqrt{v_0^{(i)}}\right)} \ . \tag{95}$$

The eigenvalues for $\rho \to \infty$ therefore approach solutions to $\sin(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) = 0$, i.e. the hyperspherical spectrum of $\tilde{\lambda} = 4n^2$. The equation to the next order in $1/\rho$ is instead

$$\tan(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) = -\frac{\sqrt{\tilde{\lambda}}}{\rho}\sum_{i=1}^{3} a_{scat}^{(i)}\mu_{jk} \ , \tag{96}$$

which immediately generalizes the symmetric case in eq. (55) into

$$\tilde{\lambda} \approx 4n^2(1 - \frac{4}{\pi\rho}\sum_{i=1}^{3} a_{scat}^{(i)}\mu_{jk}) \ . \tag{97}$$

The different patological cases of extremely large scattering lengths are very different. Without loss of generality we can here assume that $|a_{scat}^{(1)}| \leq |a_{scat}^{(2)}| \leq |a_{scat}^{(3)}|$ and consider various cases. Using eqs. (91), (93) and (94) we obtain for large values of ρ that the angular eigenvalues when $|a_{scat}^{(2)}| \ll \rho \ll |a_{scat}^{(3)}|$ still are given by $\sin(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) = 0$ and when $|a_{scat}^{(1)}| \ll \rho \ll |a_{scat}^{(2)}|$ instead are given by

$$\pm\sqrt{\tilde{\lambda}}\cos(\frac{\pi}{2}\sqrt{\tilde{\lambda}})\sin(2\varphi_1) = 2\sin\left((\varphi_1 - \pi/2)\sqrt{\tilde{\lambda}}\right) \tag{98}$$

and finally when $\rho \ll |a_{scat}^{(1)}|$ we find

$$\left(\frac{\sqrt{\tilde{\lambda}}\cos(\frac{\pi}{2}\sqrt{\tilde{\lambda}})}{2F}\right)^3 - \left(\frac{\sqrt{\tilde{\lambda}}\cos(\frac{\pi}{2}\sqrt{\tilde{\lambda}})}{2F}\right)\frac{(f_1^2 + f_2^2 + f_3^2)}{F^2} + 2 = 0 \ . \tag{99}$$

The last equation has the symmetric solution, $\sqrt{\tilde{\lambda}}\cos(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) = -4F$, when all the quantities are independent of i. These results reflect the fact that the Efimov anomaly only is present in the last two cases where at least two of the scattering lengths are infinitly large with the resulting infinitly many bound three-body states obtained from eq. (7). Clearly we are able to extend these expressions to any order of correction in $1/\rho$.

When $\tilde{\lambda}/\rho^2$ diverges we can as in the symmetric case again define ϵ_x in the large-distance limit by $\tilde{\lambda} \to 2m\epsilon_x\rho^2/\hbar^2 \equiv \bar{\epsilon_x}\rho^2$. Eq. (91) can at most have solutions for negative $\bar{\epsilon_x}$. At large ρ we have approximately

$$\sqrt{\tilde{\lambda}} \approx -i\rho\sqrt{-\bar{\epsilon_x}}\ , \ \ \kappa_i \approx \rho\sqrt{v_0^{(i)} + \bar{\epsilon_x}}\ , \ \ \alpha_0^{(i)}\sqrt{\tilde{\lambda}} \approx -iX_i\sqrt{-\bar{\epsilon_x}/2}\ ,$$
$$\frac{\kappa_i}{\sqrt{\tilde{\lambda}}} \approx i\sqrt{-(v_0^{(i)} + \bar{\epsilon_x})/\bar{\epsilon_x}}\ \ , \ \ \kappa_i\alpha_0^{(i)} \approx X_i\sqrt{v_0^{(i)} + \bar{\epsilon_x}}\ . \tag{100}$$

The dominating term in the determinat in eq. (91) is $B_1B_2B_3$ which diverges as ρ^3. To leading order in ρ we therefore obtain the eigenvalue solutions from $B_i = 0$. This is equivalent (see eqs. (E4) and (E7)) to $d_{ii} = 0$ which immediately to leading order is seen to be identical to eq. (57) for the i'th two-body subsystem. Thus we obtain the general result that there is a one to one correspondence between the total number of two-body bound states in all the subsystems and the parabolically diverging $\tilde{\lambda}-$values at large ρ.

The behavior of $P_{nn'}$ and $Q_{nn'}$ at large distance is qualitatively the same as for three identical particles. This result, as well as the various solutions described in this subsection, is valid in general for all short-range potentials. In particular, the Efimov conditions remain the same for such potentials.

V. TWO IDENTICAL SPIN-1/2 PARTICLES

We consider a system of two identical spin–1/2 particles (labeled "f" and refered to as fermions) and a third particle (labeled by "c" and called the core) of spin s_c. The total angular momentum J can then take the values $J = s_c, s_c \pm 1$, since we only consider vanishing orbital angular momentum. The spin wave functions $\chi_s^{(i)}$ from eq. (10) are $\chi_0^{(1)}$, $\chi_{s_c\pm1/2}^{(2)}$, and $\chi_{s_c\pm1/2}^{(3)}$, which are defined as in subsection 2.2. We have labeled the core as particle 1 and the two fermions as particles 2 and 3. The spin state $\chi_1^{(1)}$ is not possible due to the required antisymmetrization of the wave function under exchange of the fermions. Since we only include s-waves the two fermions can not be coupled to spin 1. The total angular momentum is therefore confined to be $J = s_c$.

The form of the Faddeev equations depends on the spin dependence of the interactions. The fermion-fermion interaction, $V_{ff}(r)$, is only effective in relative s-states and no spin dependence is needed. For the fermion–core interaction we take the spin–dependence to be of the form

$$V_2(r) = V_3(r) = (1 + \gamma_s \mathbf{s}_c \cdot \mathbf{s}_f)V_{fc}(r)\ . \tag{101}$$

This potential is diagonal in spin space, and the diagonal matrix elements for $i = 2, 3$ are:

$$\langle\chi_{s_c+1/2}^{(i)}|V_i(r)|\chi_{s_c+1/2}^{(i)}\rangle \equiv V_{fc}^+(r) = (1 + \gamma_s\frac{s_c}{2})V_{fc}(r) \tag{102}$$

$$\langle\chi_{s_c-1/2}^{(i)}|V_i(r)|\chi_{s_c-1/2}^{(i)}\rangle \equiv V_{fc}^-(r) = (1 - \frac{\gamma_s(s_c+1)}{2})V_{fc}(r)\ . \tag{103}$$

The spin–overlaps defined in eq. (15) are now found to be

$$\begin{array}{lllll} C^{12}_{0,s_c-1/2} & = & C^{13}_{0,s_c-1/2} & = & -\sqrt{\frac{s_c}{2s_c+1}}\ , \\ C^{12}_{0,s_c+1/2} & = & -C^{13}_{0,s_c+1/2} & = & \sqrt{\frac{s_c+1}{2s_c+1}}\ , \\ C^{23}_{s_c-1/2,s_c-1/2} & = & C^{23}_{s_c+1/2,s_c+1/2} & = & -\frac{1}{2s_c+1}\ , \\ C^{23}_{s_c-1/2,s_c+1/2} & = & -C^{23}_{s_c+1/2,s_c-1/2} & = & \frac{\sqrt{4s_c(s_c+1)}}{2s_c+1}\ . \end{array} \tag{104}$$

The Pauli principle requires that the solutions are antisymmetric in a simultaneous interchange of all the coordinates of the two fermions labeled 2 and 3. This means that exchange of α_2 and α_3 and exchange of the order of the couplings in the spin functions must give a change of sign of the total wave function. Imposing this constraint the components of the three-body wave function in eq. (10) must be related as

$$\phi^{(3)}_{s_c-1/2} = \phi^{(2)}_{s_c-1/2}, \qquad \phi^{(3)}_{s_c+1/2} = -\phi^{(2)}_{s_c+1/2} \, . \tag{105}$$

The corresponding Faddeev equations obtained from eq. (16) after integrating away the spin degrees of freedom are then given by

$$\begin{aligned}
&\left[-\tfrac{\partial}{\partial\alpha_1^2} - (\tilde{\lambda}(\rho) - \rho^2 v_{ff}(\rho\sin\alpha_1))\right]\phi_0^{(1)}(\rho,\alpha_1) = -\rho^2 v_{ff}(\rho\sin\alpha_1) \\
&\quad\times\left[\tfrac{1}{\sin(2\varphi)}\left(C^{12}_{0,s_c-1/2}\int_{|\varphi-\alpha_1|}^{\pi/2-|\pi/2-\varphi-\alpha_1|} d\alpha_2\phi^{(2)}_{s_c-1/2}(\rho,\alpha_2)\right.\right. \\
&\qquad\left. + C^{12}_{0,s_c+1/2}\int_{|\varphi-\alpha_1|}^{\pi/2-|\pi/2-\varphi-\alpha_1|} d\alpha_2\phi^{(2)}_{s_c+1/2}(\rho,\alpha_2)\right) \\
&\quad\left. + \tfrac{1}{\sin(2\tilde{\varphi})}\left(C^{12}_{0,s_c-1/2}\int_{|\tilde{\varphi}-\alpha_1|}^{\pi/2-|\pi/2-\tilde{\varphi}-\alpha_1|} d\alpha_3\phi^{(3)}_{s_c-1/2}(\rho,\alpha_3)\right.\right. \\
&\qquad\left.\left. - C^{12}_{0,s_c+1/2}\int_{|\tilde{\varphi}-\alpha_1|}^{\pi/2-|\pi/2-\tilde{\varphi}-\alpha_1|} d\alpha_3\phi^{(3)}_{s_c+1/2}(\rho,\alpha_3)\right)\right] \, ,
\end{aligned} \tag{106}$$

$$\begin{aligned}
&\left[-\tfrac{\partial}{\partial\alpha_2^2} - (\tilde{\lambda}(\rho) - \rho^2 v^{\mp}_{fc}(\rho\sin\alpha_2))\right]\phi^{(2)}_{s_c\mp1/2}(\rho,\alpha_2) = \\
&\quad - \rho^2 v^{\mp}_{fc}(\rho\sin\alpha_2)\left[C^{12}_{0,s_c\mp1/2}\tfrac{1}{\sin(2\varphi)}\int_{|\varphi-\alpha_2|}^{\pi/2-|\pi/2-\varphi-\alpha_2|} d\alpha_1\phi_0^{(1)}(\rho,\alpha_1) + \right. \\
&\qquad\tfrac{1}{\sin(2\tilde{\varphi})}\left(C^{23}_{s_c\mp1/2,s_c-1/2}\int_{|\tilde{\varphi}-\alpha_2|}^{\pi/2-|\pi/2-\tilde{\varphi}-\alpha_2|} d\alpha_3\phi^{(3)}_{s_c-1/2}(\rho,\alpha_3)\right. \\
&\qquad\left.\left. + C^{23}_{s_c\mp1/2,s_c+1/2}\int_{|\tilde{\varphi}-\alpha_2|}^{\pi/2-|\pi/2-\tilde{\varphi}-\alpha_2|} d\alpha_3\phi^{(3)}_{s_c+1/2}(\rho,\alpha_3)\right)\right] \, ,
\end{aligned} \tag{107}$$

where the reduced potentials, $v_{ff} = 2mV_{ff}/\hbar^2, v^{\mp}_{fc} = 2mV^{\mp}_{fc}/\hbar^2$, are defined as in the previous sections. The two equations with second derivative of α_3, analogous to those of eq. (107), turn out to be identical to eq. (107) due to the constraints in eq. (105). We have then three independent Faddeev equations.

The angles φ and $\tilde{\varphi}$ obtained from eqs. (12) and (13) are given explicitly by

$$\varphi = \arctan\left(\frac{M+2m}{M}\right)^{1/2} \, , \tag{108}$$

$$\tilde{\varphi} = \arctan\left(\frac{M(M+2m)}{m^2}\right)^{1/2} \, , \tag{109}$$

where $M = m_1$ is the mass of the core and $m = m_2 = m_3$ the mass of each of the two fermions.

We shall again use the square-well potentials for the radial shapes of the two-body potentials between both fermion and core and between the two fermions, i.e.

$$V_{ff}(r) = -V_0^{(ff)}\Theta(r < R_{ff}) \, , \tag{110}$$

$$V_{fc}(r) = -V_0^{(fc)}\Theta(r < R_{fc}) \, . \tag{111}$$

The related reduced potentials are then defined as

$$v_{ff}(x) = -v_0^{(ff)}\Theta(x < X_{ff}) \, , \tag{112}$$

$$v_{fc}(x) = -v_0^{(fc)}\Theta(x < X_{fc}) \, , \tag{113}$$

and the depth parameters for the two fermion-core relative spin states are

$$v_0^{(fc+)}(x) = -v_0^{(fc)}(1 + \gamma_s\frac{s_c}{2}) \, , \tag{114}$$

$$v_0^{(fc-)}(x) = -v_0^{(fc)}(1 - \frac{\gamma_s(s_c+1)}{2}) \, , \tag{115}$$

where $X_{fc} = R_2\mu_{13}$ and $X_{ff} = R_1\mu_{23}$ are defined as in section 4.

We shall also in this section restrict ourselves to small or large distances and omit the lengthy but straightforward calculations and expressions for the less interesting intermediate distances.

A. Short-distance behavior: $0 \leq \rho \leq \mathrm{Min}(X_{ff}, X_{fc})$

The potentials are again constants for all α–values and, as in subsection 4.1, eqs. (106) and (107) have the general solutions

$$\phi_s^{(i)}(\rho, \alpha_i) = a_{s,i} \sin(2n\alpha_i) \ , \tag{116}$$

where the constants $a_{0,1}$, $a_{s_c\pm 1/2,2}$, and $a_{s_c\pm 1/2,3}$ are found by substitution of eq. (116) into eqs. (106) and (107). We obtain the following system of linear equations for

$$\begin{aligned} &(\epsilon - v_0^{(ff)}) a_{0,1} = \\ &- v_0^{(ff)} \left[dC^{12}_{0,s_c-1/2}(a_{s_c-1/2,2} + a_{s_c-1/2,3}) + dC^{12}_{0,s_c+1/2}(a_{s_c+1/2,2} - a_{s_c+1/2,3}) \right] \ , \end{aligned} \tag{117}$$

$$\begin{aligned} &(\epsilon - v_0^{(fc\mp)}) a_{s_c\mp 1/2,2} = \\ &- v_0^{(fc\mp)} \left[dC^{12}_{0,s_c\mp 1/2} a_{0,1} + \tilde{d}(C^{23}_{s_c\mp 1/2,s_c-1/2} a_{s_c-1/2,3} + C^{23}_{s_c\mp 1/2,s_c+1/2} a_{s_c+1/2,3}) \right] \ , \end{aligned} \tag{118}$$

where we in analogy with eq. (78) define

$$\tilde{\lambda}(\rho) = 4n^2 - \rho^2\epsilon \ , \qquad d = -\frac{1}{n}\frac{\sin(2n\varphi)}{\sin(2\varphi)} \ , \qquad \tilde{d} = -\frac{1}{n}\frac{\sin(2n\tilde{\varphi})}{\sin(2\tilde{\varphi})} \ . \tag{119}$$

The antisymmetry expressed in eq. (105) relate the unkown constants by

$$a_{s_c-1/2,3} = a_{s_c-1/2,2} \ , a_{s_c+1/2,3} = -a_{s_c+1/2,2} \ , \tag{120}$$

which by use in eqs. (117) and (118) reduces the number of equations and the number of unknown constants to three.

The solutions for ϵ or $\tilde{\lambda}(\rho)$ are now obtained by demanding non-vanishing solutions for the constants $a_{s,i}$. This amounts effectively to three linear equations and the corresponding determinant must vanish. Thus ϵ is obtained from

$$\begin{vmatrix} \epsilon - v_0^{(ff)} & 2v_0^{(ff)} dC^{12}_{0,s_c-1/2} & 2v_0^{(ff)} dC^{12}_{0,s_c+1/2} \\ v_0^{(fc-)} dC^{12}_{0,s_c-1/2} & \epsilon - v_0^{(fc-)}(1 - \tilde{d}C^{23}_{s_c-1/2,s_c-1/2}) & -v_0^{(fc-)} \tilde{d} C^{23}_{s_c-1/2,s_c+1/2} \\ v_0^{(fc+)} dC^{12}_{0,s_c+1/2} & -v_0^{(fc+)} \tilde{d} C^{23}_{s_c-1/2,s_c+1/2} & \epsilon - v_0^{(fc+)}(1 + \tilde{d}C^{23}_{s_c-1/2,s_c-1/2}) \end{vmatrix} = 0 \ . \tag{121}$$

This means that there are at most three λ-solutions for each n; sometimes less than three, since some of them can be the trivial spurious solutions characterized by $\epsilon = 0$.

When the determinant vanishes for $\epsilon = 0$, we find that these spurious solutions must satisfy one or both of the following two conditions:

$$n - \frac{\sin(4n\varphi)}{\sin(4\varphi)} = 0 \ , \tag{122}$$

$$n^2 + \frac{\sin(4n\varphi)}{\sin(4\varphi)} n - 2\frac{\sin^2(2n\varphi)}{\sin^2(2\varphi)} = 0 \ . \tag{123}$$

For $n = 1$ ($d = \tilde{d} = -1$) these two equations are both satisfied, meaning that two of the solutions are spurious and only one antisymmetric solution exists. Solving the determinant (121) the antisymmetric solution is found to be

$$\tilde{\lambda}(\rho) = \lambda(\rho) + 4 = 4 - \rho^2 \left(v_0^{(ff)} + v_0^{(fc-)} \frac{2s_c}{2s_c+1} + v_0^{(fc+)} \frac{2s_c+2}{2s_c+1} \right) , \tag{124}$$

which reduces to eq. (77) for identical and spin independent potentials.

For $n = 2$, where $2d^2 = 1 - \tilde{d}$, only the condition in eq. (123) is satisfied, and two non-spurious antisymmetric solutions appear. They are the solutions of the second order equation

$$\epsilon^2 - \epsilon \left(v_0^{(ff)} + v_0^{(fc+)} + v_0^{(fc-)} + \frac{\tilde{d}}{2s_c+1}(v_0^{(fc-)} - v_0^{(fc+)}) \right) \tag{125}$$
$$+ (1+\tilde{d}) \left(v_0^{(ff)} v_0^{(fc-)} \frac{s_c+1}{2s_c+1} + v_0^{(ff)} v_0^{(fc+)} \frac{s_c}{2s_c+1} + v_0^{(fc+)} v_0^{(fc-)} (1-\tilde{d}) \right) = 0 \, ,$$

which combined with eq.(119) results in two solutions for $\tilde{\lambda}(\rho)$.

For $n \geq 3$ none of the conditions (122) and (123) are satisfied, and three non-spurious antisymmetric solutions are found. For very big values of n both d and $\tilde{d}$ approach zero, and the three solutions of the determinant converges towards $\epsilon = v_0^{(ff)}$, $\epsilon = v_0^{(fc-)}$ and $\epsilon = v_0^{(fc+)}$.

B. Large-distance behavior

As in eqs.(30) and (74), we define

$$\alpha_0^{(ff)} = \arcsin(X_{ff}/\rho) \, , \tag{126}$$
$$\alpha_0^{(fc)} = \arcsin(X_{fc}/\rho) \tag{127}$$

such that the potential is non-vanishing only when the corresponding $\alpha \leq \alpha_0$.

Since $\alpha_0^{(ff)}$ and $\alpha_0^{(fc)}$ approach zero for increasing ρ, we can define large distances by

$$\alpha_0^{(ff)} \leq \alpha_0^{(fc)} \leq |\varphi - \alpha_0^{(ff)}| \leq |\varphi - \alpha_0^{(fc)}| \, , \tag{128}$$
$$\alpha_0^{(ff)} \leq \alpha_0^{(fc)} \leq |\tilde{\varphi} - \alpha_0^{(ff)}| \leq |\tilde{\varphi} - \alpha_0^{(fc)}| \, , \tag{129}$$

where we assumed that $X_{ff} < X_{fc}$.

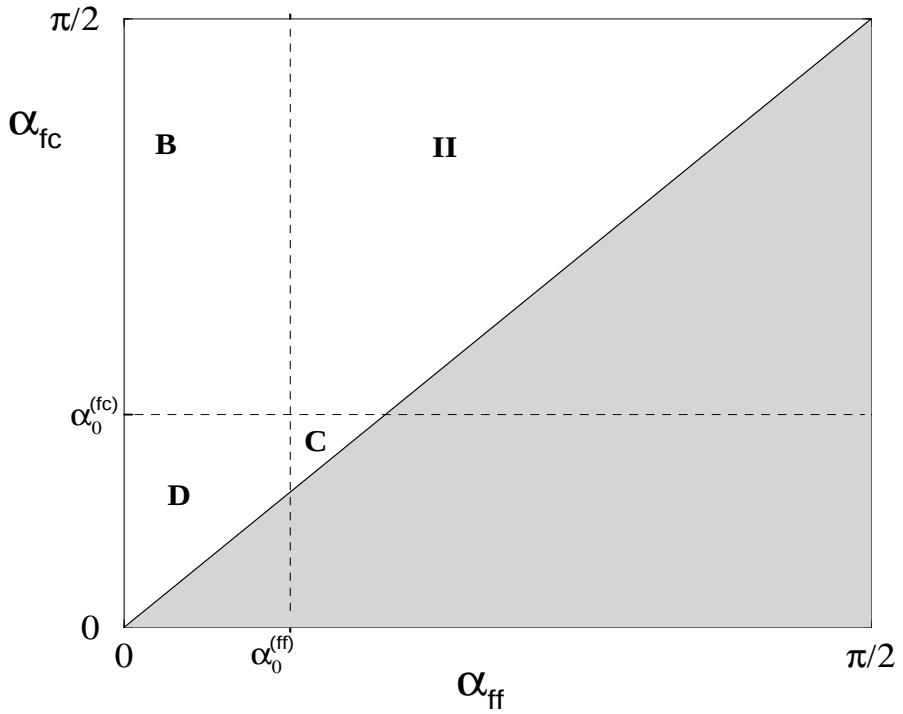

FIG. 4. The regions, defined in subsection 5.2, in $(\alpha_{fc}, \alpha_{ff})-$space arising for large distances for two identical spin-1/2 particles. The shaded area does not enter in the present computation.

In fig. 4 we plot $\alpha_{fc} = \alpha_2$ versus $\alpha_{ff} = \alpha_1$ where only the region, $\alpha_{ff} < \alpha_{fc}$, is relevant for our calculation. In the plot we can distinguish four different zones:

$$\begin{aligned}
II &: \alpha_{fc} > \alpha_0^{(fc)} \text{ and } \alpha_{ff} > \alpha_0^{(ff)} \Longrightarrow V_{fc}(\rho \sin\alpha_{fc}) = 0 \text{ and } V_{ff}(\rho \sin\alpha_{ff}) = 0 \\
B &: \alpha_{fc} > \alpha_0^{(fc)} \text{ and } \alpha_{ff} < \alpha_0^{(ff)} \Longrightarrow V_{fc}(\rho \sin\alpha_{fc}) = 0 \text{ and } V_{ff}(\rho \sin\alpha_{ff}) \neq 0 \\
C &: \alpha_{fc} < \alpha_0^{(fc)} \text{ and } \alpha_{ff} > \alpha_0^{(ff)} \Longrightarrow V_{fc}(\rho \sin\alpha_{fc}) \neq 0 \text{ and } V_{ff}(\rho \sin\alpha_{ff}) = 0 \\
D &: \alpha_{fc} < \alpha_0^{(fc)} \text{ and } \alpha_{ff} < \alpha_0^{(ff)} \Longrightarrow V_{fc}(\rho \sin\alpha_{fc}) \neq 0 \text{ and } V_{ff}(\rho \sin\alpha_{ff}) \neq 0
\end{aligned}$$

Since $\alpha_0^{(ff)}$ and $\alpha_0^{(fc)}$ are close to zero, regions B, C, and D are very small. Due to the inequalities in eqs.(128) and (129), the integrations appearing in the Faddeev equations involve only the functions in the region where the potentials vanish.

The wave functions must vanish at $\alpha = 0$ and $\alpha = \pi/2$ and the solutions in region II, where all potentials are zero, are therefore of the form

$$\phi_s^{(i)}(\alpha_i) = b_{s,i} \sin\left((\alpha_i - \pi/2)\sqrt{\tilde{\lambda}}\right) \ . \tag{130}$$

In the other regions (B,C and D) the same form of the wave function is a solution when the corresponding potentials vanish, i.e. for $\alpha_i > \alpha_0^{(i)}$. When $\alpha_i < \alpha_0^{(i)}$ in these regions, we have solutions of the form

$$\phi_s^{(i)}(\alpha_i) = c_{s,i} \sin(\alpha_i \kappa_s) + d_{s,i} \sin\left(\alpha_i \sqrt{\tilde{\lambda}}\right) \ , \tag{131}$$

where the first term is the solution to the homogeneous equation and consequently the different κ_s-values are given by

$$\kappa_0 = \sqrt{v_0^{(ff)}\rho^2 + \tilde{\lambda}(\rho)} \ , \quad \kappa_{s_c \pm 1/2} = \sqrt{v_0^{(fc\pm)}\rho^2 + \tilde{\lambda}(\rho)} \ . \tag{132}$$

The constants $c_{s,i}$ are therefore completely free, since the functions to be integrated in eqs.(106) and (107) only involve the functions in eq.(130), i.e. values of $\alpha_i > \alpha_0^{(i)}$ for all three components, see fig. 4. The constants $b_{s,i}$ and $d_{s,i}$ are linearly related through the Faddeev equations.

The detailed solutions in the different regions are given in appendix F. The eigenvalue equation again takes the form of a vanishing determinant where the matrix elements are given in eq.(F13).

In the limit of very large distances we can for finite $\tilde{\lambda}-$values use an expansion to lowest order in $1/\rho$ as given in appendix F, where also the case of diverging $\tilde{\lambda}-$values are considered. The linear set of equations in eq.(F12) then reduces to

$$A_1 \left(\sqrt{\tilde{\lambda}} \cos(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) \frac{a_{scat}^{(ff)}}{\rho} \mu_{23} + \sin(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) \right) = -a_0 \sqrt{\tilde{\lambda}} \frac{a_{scat}^{(ff)}}{\rho} \mu_{23} \tag{133}$$

$$A_2 \left(\sqrt{\tilde{\lambda}} \cos(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) \frac{a_{scat}^{(fc-)}}{\rho} \mu_{12} + \sin(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) \right) = -a_{s_c-1/2} \sqrt{\tilde{\lambda}} \frac{a_{scat}^{(fc-)}}{\rho} \mu_{12} \tag{134}$$

$$A_3 \left(\sqrt{\tilde{\lambda}} \cos(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) \frac{a_{scat}^{(fc+)}}{\rho} \mu_{12} + \sin(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) \right) = -a_{s_c+1/2} \sqrt{\tilde{\lambda}} \frac{a_{scat}^{(fc+)}}{\rho} \mu_{12} \ , \tag{135}$$

where μ_{12} and μ_{23} are constants defined in appendix A, and $a_0, a_{s_c \pm 1/2}$ depend linearly on the coefficients A_i as defined in appendix F. The eigenvalues for $\rho \to \infty$ therefore for finite scattering lengths approach solutions to $\sin(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) = 0$, i.e. the hyperspherical spectrum of $\tilde{\lambda} = 4n^2$.

The solutions to the next order in $1/\rho$ only exist when the corresponding determinant is zero which then defines the behavior of $\tilde{\lambda}(\rho)$ for large values of ρ. We have previously assumed that $\alpha_0^{(ff)} < \alpha_0^{(fc)}$ and both quantities are close to zero in the large-distance limit. If we further assume that $\alpha_0^{(ff)} = 0$ or equivalently $X_{ff} = 0$, only regions II and C survive in fig.4. Then $A_1 = 0$, $|a_{scat}^{(ff)}|/\rho \approx 0$ (for finite $a_{scat}^{(ff)}$ and large ρ) and the fermion-fermion interaction completely disappears. The determinant is reduced to a 2 by 2 determinant, that gives the following expression for $\tilde{\lambda}(\rho)$

$$\frac{\rho^2}{a_{scat}^{(fc+)}a_{scat}^{(fc-)}}\sin^2\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)+\frac{1}{2}\sqrt{\tilde{\lambda}}\sin(\pi\sqrt{\tilde{\lambda}})\mu_{12}\left(\frac{\rho}{a_{scat}^{(fc+)}}+\frac{\rho}{a_{scat}^{(fc-)}}\right)$$
$$+\frac{2}{2s_c+1}\left(\frac{\rho}{a_{scat}^{(fc-)}}-\frac{\rho}{a_{scat}^{(fc+)}}\right)\tilde{f}\sin\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right) \qquad (136)$$
$$+\tilde{\lambda}\cos^2\left(\frac{\pi}{2}\sqrt{\tilde{\lambda}}\right)\mu_{12}^2-4\mu_{12}^2f^2=0\ ,$$

where f and $\tilde{f}$ are defined in eq.(F5).

In the limit where both scattering lengths are large compared to ρ, i.e. $|a_{scat}^{(fc\pm)}| \gg \rho$, we arrive again at eq.(98). The occurrence conditions and properties of the Efimov states can be derived from eq.(136). The details are discussed in [19].

The behavior of $P_{nn'}$ and $Q_{nn'}$ at large distance is qualitatively the same as for three spinless particles. This result is again, along with the various solutions described in this subsection, valid in general for all short-range potentials. In particular, also the Efimov conditions remain the same for such potentials.

VI. NUMERICAL ILLUSTRATIONS AND GENERALIZATIONS

The analytical solutions in the previous sections are derived for square-well potentials. The results are similar for other short-range potentials. Illustrations by use of smoother gaussian potential are therefore appropriate and we shall first show numerically calculated angular eigenvalue spectra for different symmetries of the three-body system. Then we shall give a survey of the analytical results for different distances and show how the results can be used for arbitrary short-range potentials. The simplest case of intrinsic spins of the particles were also considered in the previous sections. Other cases with more complicated spin structures can be worked out. It is straightforward, but results easily in rather extended formulae. We shall here indicate how to proceed in the case of two different spin-1/2 particles.

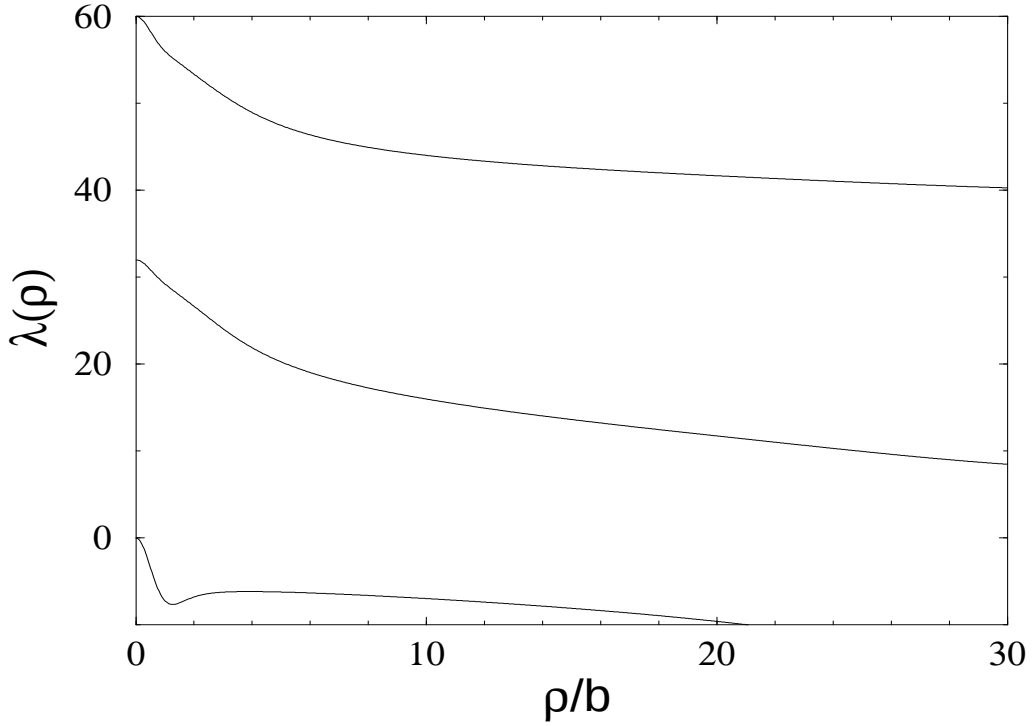

FIG. 5. The lowest angular eigenvalues $\lambda = \tilde{\lambda} - 4$, when all orbital angular momenta are zero, are shown as function of ρ/b for the case of three identical bosons for a gaussian potential, $\bar{V}_0\hbar^2/mb^2$ exp(-$(r/b)^2$), where m is the mass of the bosons. The actual value of $\bar{V}_0 = -3.08$ corresponds to a bound two-body state.

A. Numerical illustrations

The angular eigenvalues, still only s-states, for the symmetric case of three identical bosons are shown in fig. 5 as function of ρ/b for a gaussian potential. The parameters of the potential correspond to a two-body bound state. The lowest eigenvalue consequently bends over and diverges parabolically as described in eq.(56). The higher lying levels then come down and the hyperspherical spectrum seen at $\rho = 0$ is approached at large distances. The values of $\lambda = 4n^2 - 4$ at $\rho = 0$ are 0, 32, 60 and the spurious state starting from $\lambda = 12$ corresponding to $n = 2$ in eq.(29) is omitted. Due to the symmetry requirement only one state appears for each value of n.

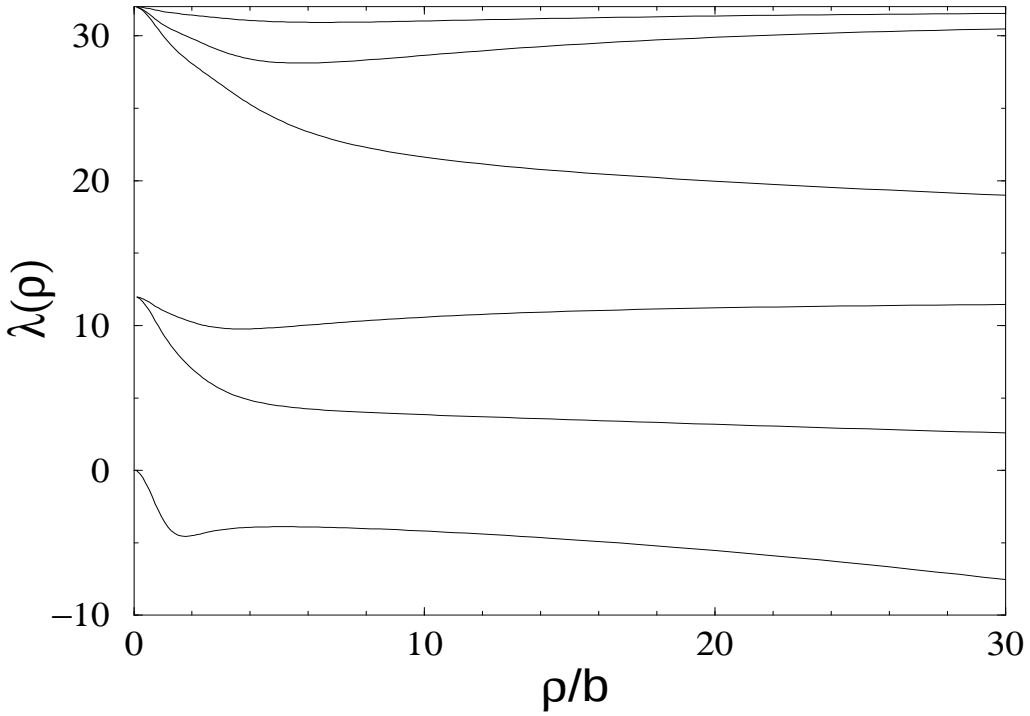

FIG. 6. The lowest angular eigenvalues $\lambda = \tilde{\lambda} - 4$, when all orbital angular momenta are zero, are shown as function of ρ/b for three different spinless particles for gaussian potentials, $\bar{V}_0^{(i)}\hbar^2/m_i b^2 \exp(-(r/b)^2)$. The actual values $\bar{V}_0^{(i)} = -1.28, -1.54, -3.08$ and $m_i = m_1, 2m_1, 3m_1$, respectively for $i = 1, 2, 3$, correspond to a bound two-body state in the subsystem labeled i=1 and no bound states in the other subsystems.

In fig. 6 we show the angular eigenvalues as function of ρ/b, again only s-states, for the asymmetric case of three different spinless particles interacting via gaussian potentials. Only one of the potentials has a bound two-body state and consequently the lowest eigenvalue bends over and diverges parabolically as described in eq.(100). Again the higher lying levels come down and the spectrum at $\rho = 0$ emerges also in this case at large distances. The spurious level corresponding to n=2 is still omitted, but now two other levels appear at $\lambda = 12$ corresponding to (spatially) non-symmetric configurations. Also for n=3 at $\lambda = 32$ two more levels corresponding to non-symmetric states appear in addition to the totally symmetric state. For all values of $n \geq 3$ the structure of one symmetric and two asymmetric states remains unchanged.

In fig. 7 we show analogous numerical results for two identical spin-1/2 fermions plus one third particle (core) of spin $s_c = 3/2$. Now one additional degree of freedom appears. It is related to the two different fermion-core spin couplings. However, simultaneously the two fermions are restricted to totally antisymmetric two-body states. This clearly removes some of the otherwise possible states. Since the hyperspherical spectrum always both is the starting point at $\rho = 0$ and the asymptotic limit for large ρ, the combined result is a spectrum very similar to that of fig. 6 corresponding to three different particles without spin degrees of freedom. If the core had been spinless, the fermion-core spin coupling had been unique and one state less would have appeared for every value of $n \geq 2$.

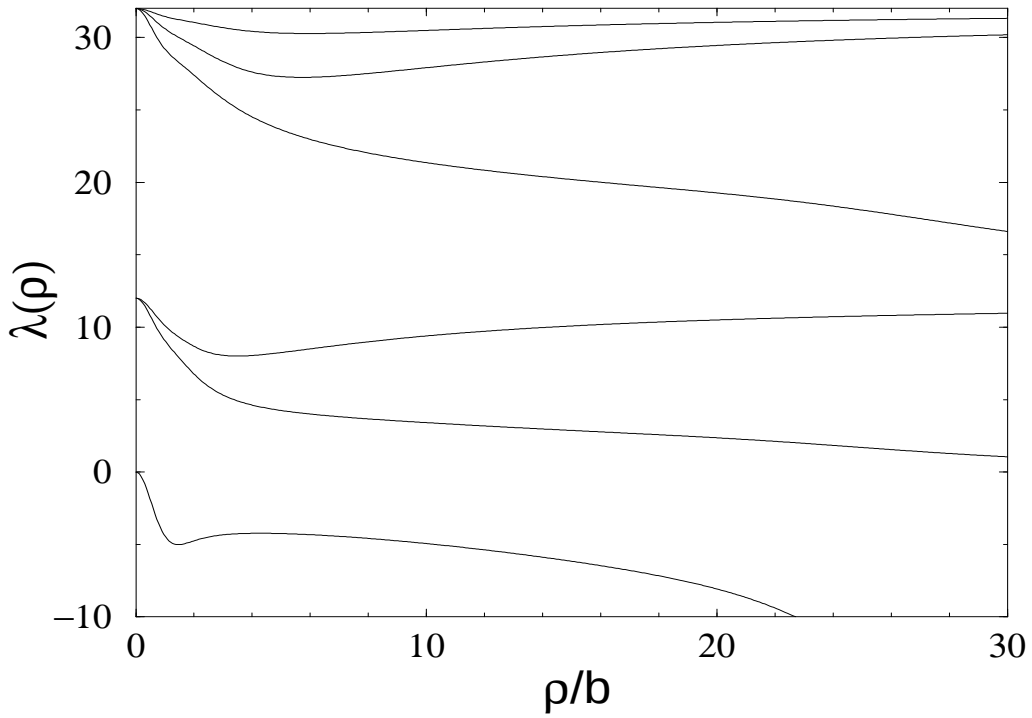

FIG. 7. The lowest angular eigenvalues $\lambda = \tilde{\lambda} - 4$, when all orbital angular momenta are zero, are shown as function of ρ/b for gaussian potentials, $\bar{V}_0^{(i)}\hbar^2/m_i b^2$ exp(-$(r/b)^2$), for a system of two identical spin-1/2 fermions plus a third particle of spin $s_c = 3/2$. The actual values $\bar{V}_0^{(i)} = -6.16, -0.99, -1.09$ and $m_i = 2m_2, m_2, m_2$, respectively for $i = 1, 2, 3$, correspond to the fermion-fermion interaction, the fermion-core interaction for relative spin of 2 and the fermion-core interaction for relative spin of 1, respectively for $i = 1, 2$ and 3.

B. Survey and generalization of the solutions

Various analytical solutions apply to different regions in the two-dimensional (ρ, α) coordinate space. A survey is shown in fig. 8 where the regions labeled I,A,B,C and II refer to the division in the subsections of the symmetric case. (The more general asymmetric case requires more divisions at intermediate distances. However, the picture basically remains unchanged.) The simplest cases of small and large distances (I and II) actually constitute a very large fraction of the total coordinate space. These solutions are also appropriate for general short-range potentials where the approximate validity results from perturbative treatments.

At large distances, where α_0 and κ can be expanded to first order in $1/\rho$, we obtain considerable further simplification. The eigenvalue equation can in this limit be expressed entirely in terms of the scattering length of the potential. In fig. 9 we show the lowest angular eigenvalue for a square-well potential as function of ρ. We compare with the approximate solution at very large distances given in eq.(55). The $1/\rho$ asymptotic convergence towards the limit is clearly seen. This approximation is normally too inaccurate for the interesting shorter distances.

A numerical procedure for general short-range potentials now suggest itself. First we construct the equivalent square-well potential with the same scattering length and effective range as the potential in question. The solutions are then obtained analytically in $\alpha-$space as described in this paper. They can be anticipated to be very accurate solutions except in smaller regions at intermediate distances, where direct numerical integration interpolating between the available analytical solutions then must be used. The gain in accuracy and speed is substantial.

An example is shown in fig. 9 where we used an attractive gaussian potential of range b and strength $-2.19\hbar^2/mb^2$ with a scattering length, $a_{scat} = 5b$, and an effective range, $R_{eff} = 1.64b$, where b is the range of the potential. The corresponding radius and depth of the square-well potential is then $R_0 = 1.47b$ and $V_0 = -0.93\hbar^2/mb^2$, respectively.

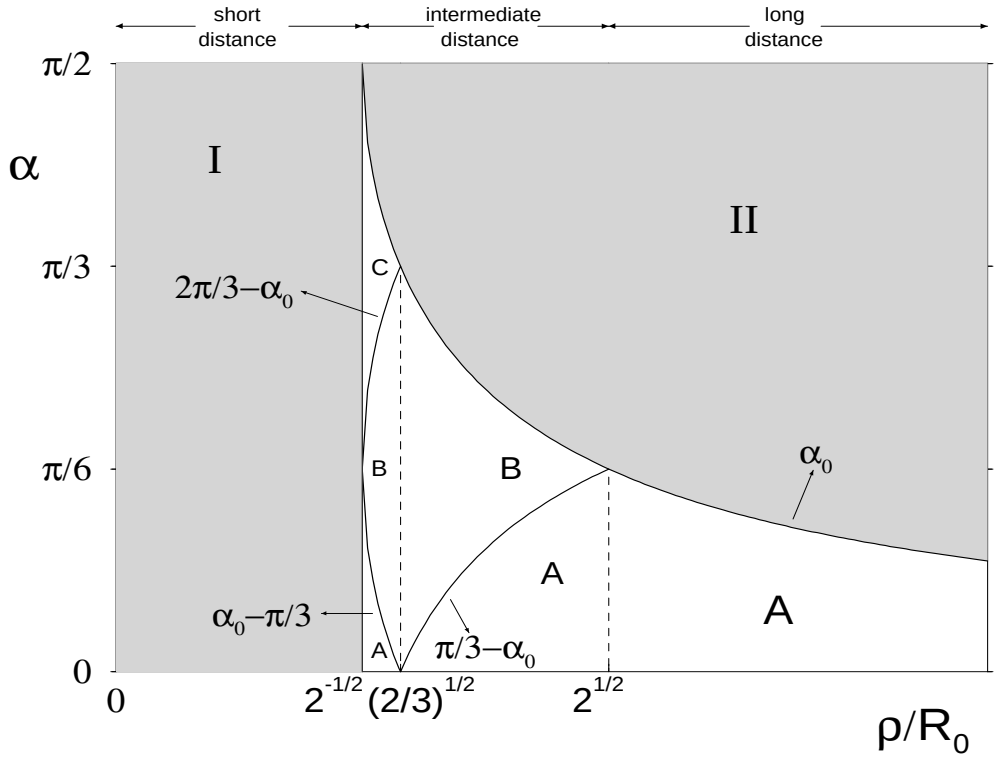

FIG. 8. The regions in α-space as function of ρ/R_0 for a square-well potential of radius R_0 for three identical bosons. The shaded areas indicate the small and large-distance regions I and II, where the wave functions are particularly simple.

The curves for the two potentials are indistinguishable at large distances down to $\rho = R_0\sqrt{2} \approx 2b$. Then the square-well result is lowest until $\rho \approx b$, where the curves cross and the gaussian result stays as the lowest all the way down to zero. The largest deviation is less than about 0.75 compared to the minimum value of about -6. This relatively small region, $0.3 \leq \rho/b \leq 2$, at intermediate distances, corresponds to the surface region for the two-body potential. The perturbative short-distance solution for the gaussian potential is accurate up to about $\rho \approx 0.3b$. The large-distance behavior describing the approach towards the asymptotic limit is only a reasonable approximation at very large distances.

The exact square-well solution is almost quantitatively a good approximation. The two energies would be close due to the similarities and the fact that a larger potential at smaller distance is compensated by a smaller potential at larger distance. The square well tends to confine the wave function in a somewhat more narrow region around the minimum.

The (smaller) differences between the two potentials occur in an essential region, where a substantial part of the attractive pocket of the effective three-body potential is contained. High accuracy for an arbitrary potential therefore requires a treatment better than that corresponding to the square-well solution. In practise the most efficient procedure is numerical integration starting with the small-distance perturbative solution. Reaching large distances corresponding to $\rho \leq R_0\sqrt{2}$ the exact square-well solution, which then is particularly simple, can be used to a very high accuracy.

C. Two different spin-1/2 particles

We consider a system consisting of two non-identical spin-1/2 particles and a third particle of arbitrary spin s_c. The procedure is then the same as in section 5, but now the spin state $\chi_1^{(1)}$ is allowed and the antisymmetrization constraint eq.(105) is not required. The three Faddeev equations can then be written as a system of six equations, where the angles φ_1, φ_2, and φ_3 all are different. Since the two fermions can couple to spin 1, the total spin of the three-body system can be $J = s_c, s_c \pm 1$. In case when $J = s_c \pm 1$, the spin states $\chi_0^{(1)}$, $\chi_{s_c \mp 1/2}^{(2)}$, and $\chi_{s_c \mp 1/2}^{(3)}$ are not possible and the total wave function is symmetric under exchange of the two fermions, and the number of Faddeev equations is again reduced to three.

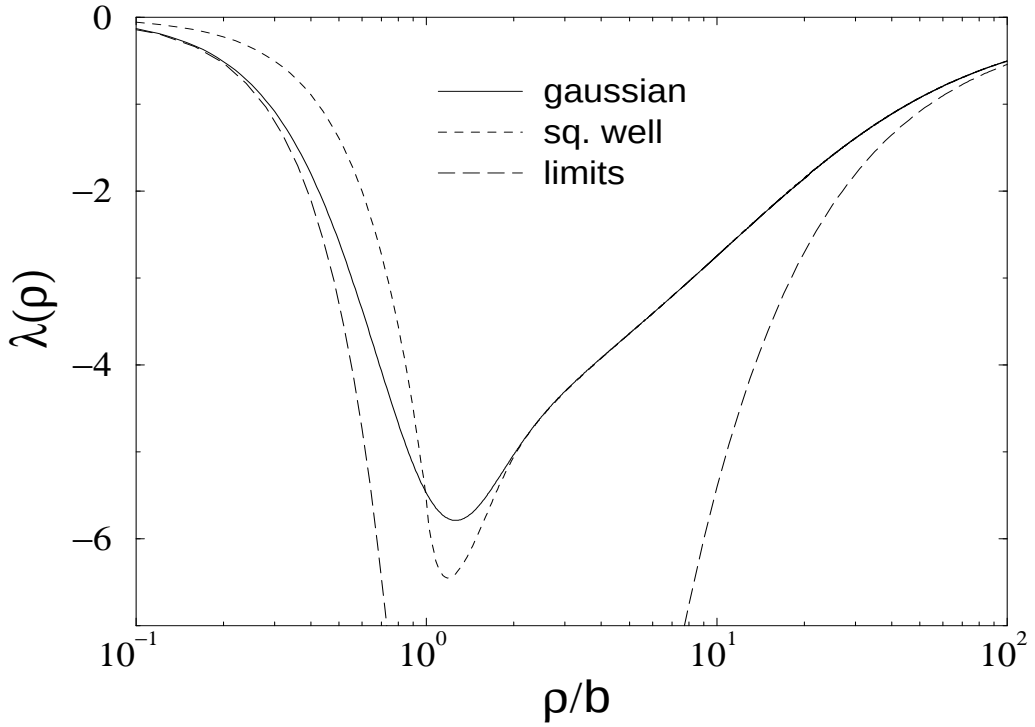

FIG. 9. The lowest angular eigenvalue $\lambda = \tilde{\lambda} - 4$ as function of ρ/b for a system of three identical bosons. The interactions correspond to a gaussian potential (solid curve), $\bar{V}_0\hbar^2/mb^2$ exp(-$(r/b)^2$), $\bar{V}_0 = -2.19$, and a square-well potential (short-dashed curve), radius $R_0 = 1.47b$ and depth $-0.93\hbar^2/mb^2$, with the same scattering length $a_{scat} = 5b$, and effective range $R_{eff} = 1.64b$. The perturbative solution for the gaussian potential at small distance (eq. (29)) and the limiting behavior at large distance (eq. (55)) are also shown (long-dashed curves).

At short-distances the general solutions are given by eq.(116). They give a 6 by 6 determinant when $J = s_c$ and a 3 by 3 determinant when $J = s_c \pm 1$. Making these determinants equal to zero and solving the equation for $\epsilon = 0$ one can extract the spurious solutions and the non-spurious ones for $\epsilon \neq 0$.

At large distances the procedure is also analogous to the one shown in the subsection 5.2. The only difference is that now we have $\alpha_0^{(f_1 f_2)}$, $\alpha_0^{(f_1 c)}$, and $\alpha_0^{(f_2 c)}$, and the number of regions we have to consider (similar to regions II, A, B, and C in subsection 5.2) is much larger. Technically the problem is more cumbersome, but conceptually it is identical.

VII. SUMMARY AND CONCLUSIONS

The new method to solve the Faddeev equations in coordinate space has several advantages especially in the long-distance description. The reason is the access to the asymptotic analytic properties made possible by the two-step sequential procedure. The generalized angular equations are first solved for each average distance (radial coordinate) between the three particles. The periodic behavior or the finite intervals characteristic for angular variables provide discrete quantized solutions which subsequently are used as a complete basis set in an expansion of the total wave function. The expansion coefficients (as well as the basis) are functions of the radial coordinate. They are in the next step determined together with the total energy from the coupled set of radial equations. We have first formulated this general procedure for s-states with the inclusion of intrinsic spins for each of the three particles. For spin-independent interactions we recover the previous general s-state Faddeev equations.

We study the symmetric case described by three equal Faddeev components. We assume spin independent interactions and arrive at one integro–differential angular Faddeev equation for s-states. This equation is solved analytically for a square-well potential. The angular wave functions are especially simple at small (one sine function) and large distances (one or two sine functions). These solutions are, with an appropriate interpretation, more general than their origin as square-well solutions seems to suggest. At small distance, they are the first order perturbative solutions for

an arbitrary potential with a value at the center equal to the square-well depth. At sufficiently large distances where the general short-range potential has vanished the solution obviously is identical to the square-well solutions outside its radius. We show explicitly that the coupled set of radial equations decouple at large distances.

At intermediate distances the solutions are linear combinations of up to four sine functions of varying arguments. The corresponding eigenvalues are explicitly given at small distance where they vary parabolically with distance starting from the hyperspherical spectrum. At intermediate distances the eigenvalues are solutions to cubic equations and a simple trancendental equation involving trigonometric functions arises at large distances.

The length scale is here defined by the radius of the square-well potential and the large-distance solution is valid when the average radius exceeds $R_0\sqrt{2}$. Expansion of the eigenvalue equation to lowest order in the inverse distance brings additional simplification, but introduces also the two-body scattering length as another length parameter. The Efimov condition leading to infinitly many bound three-body states is then obtained and seen to be independent of the short-range potential.

The asymmetric case, still s-states and spin independent interactions, is described by three Faddeev equations. We can again solve the angular part analytically for square-well potentials acting betweeen each pair of particles. Each Faddeev component of the wave functions is at small and large distances of the same form as in the symmetric case. These solutions are again appropriately interpreted solutions to general potentials either in perturbation theory or asymptotically at large distances. The corresponding eigenvalues are in this case obtained either as solutions to a cubic equation or to a trancendental equation involving trigonometric functions. At intermediate distances the solutions can still be found analytically to be combinations of exponentials and trigonometric functions. However, the number of terms is now substantial and we abstain from writing down these soulutions.

Expanding the eigenvalue equation to first order in the inverse distance again brings additional simplifications and introduces the three (different) scattering lengths. The Efimov conditions leading to infinitly many bound three-body states can then be seen to exist when at least two scattering lengths are infinitly large. Asymptotic large-distance behavior of the effective radial potentials are derived to first order in the inverse distance.

The case of two identical spin-1/2 fermions and one additional core particle of arbitrary spin can of course also be described by three Faddeev equations. We still include only s-states and assume a spin-spin type of interaction between fermion and core particles. Elimination of the spin degrees of freedom results in three angular Faddeev equations where the reduction from six to three equations arise due to the requirement of an antisymmetric wave function. Again these equations are solved for radial square-well potentials both for small and large distances. Expansion to first order in the inverse distance expresses the angular eigenvalue equation in terms of three (different) scattering lengths. The Efimov conditions are described and discussed.

We finally considered qualitatively the case of two different spin-1/2 particles interacting mutually and with a third particle. The original three Faddeev equations results now, still only including s-states, in sixcoupled angular equations after elimination of the spin degrees of freedom. Small and large-distance behavior can be studied analogously for radial square-well potentials. The systems considered here containing spin-1/2 particles are chosen as the simplest examples of three interacting particles with an intrinsic structure showing up in the form of spin degrees of freedom. Other similar examples can be worked out in the same way.

The angular solutions obtained with square-well potentials have properties characteristic for short-range potentials. To calculate the total solution we must in addition solve the coupled set of radial differential equations. This is a rather simple numerical problem. The coupled set of radial equations decouple at large distances. Only s-waves are included in the discussion, but they are usually the most interesting components in the wave functions. For bound states only s-waves extend far beyond the radii of the short-range potentials. They decouple from higher angular momentum components at large distances and often constitute the dominating part of the total wave function.

The square-well solutions are intrinsically interesting. Furthermore, they approach at large distance the solutions for arbitrary short-range potentials. The exact solutions are valid down to distances close to the radius of the square-well potentials. The solutions for general potentials can now be found by adjusting depth and radius of a square-well potential to obtain the same scattering length and effective range. The large-distance solutions obtained as described in this paper are then correct down to $\sqrt{2}$ times the radius of the square-well potential, i.e. at much smaller distances than given by the large-distance expansion for general potentials. Therefore these results can be used to make the numerical computations both substantially faster and more accurate.

In conclusion, we have discussed a method and derived solutions providing basic insight into the Faddeev equations. Practical numerical improvements are suggested for arbitrary short-range potentials.

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Appendix A: Coordinates

We consider a system of three particles with masses m_i and coordinates $\mathbf{r}_i$. The Jacobi coordinates are defined as [2,8].

$$\begin{gathered}\mathbf{x}_i = \mu_{jk}\mathbf{r}_{jk}, \mathbf{y}_i = \mu_{(jk)i}\mathbf{r}_{(jk)i} \\ \mu_{jk} = \left(\frac{1}{m}\frac{m_j m_k}{m_j + m_k}\right)^{1/2}, \mu_{(jk)i} = \left(\frac{1}{m}\frac{(m_j + m_k)m_i}{m_1 + m_2 + m_3}\right)^{1/2} \\ \mathbf{r}_{jk} = \mathbf{r}_j - \mathbf{r}_k, \mathbf{r}_{(jk)i} = \frac{m_j\mathbf{r}_j + m_k\mathbf{r}_k}{m_j + m_k} - \mathbf{r}_i \ .\end{gathered} \tag{1}$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$ and μ^2 are the reduced masses of the subsystems in units of an arbitrary normalization m.

The hyperspherical variables are introduced as

$$\rho,\ \mathbf{n}_{x_i} = \mathbf{x}_i/|\mathbf{x}_i|,\ \mathbf{n}_{y_i} = \mathbf{y}_i/|\mathbf{y}_i|,\ \alpha_i, \tag{2}$$

where α_i is in the interval $[0, \pi/2]$

$$\rho^2 = \mathbf{x_i}^2 + \mathbf{y_i}^2,\ |\mathbf{x_i}| = \rho\sin\alpha_i,\ |\mathbf{y_i}| = \rho\cos\alpha_i \ . \tag{3}$$

We omit the indices where we need not emphasize the particular set of Jacobi coordinates. Note that ρ is independent of what set is used.

The relation between three different sets of Jacobi coordinates is given by

$$\mathbf{x}_k = \mathbf{x}_i\cos\varphi_{ik} + \mathbf{y}_i\sin\varphi_{ik},\ \mathbf{y}_k = -\mathbf{x}_i\sin\varphi_{ik} + \mathbf{y}_i\cos\varphi_{ik} \tag{4}$$

where the transformation angle φ_{ik} is given by the masses as

$$\varphi_{ik} = \arctan\left((-1)^p\sqrt{\frac{m_j(m_1 + m_2 + m_3)}{m_k m_i}}\right) \tag{5}$$

where $(-1)^p$ is the parity of the permutation $\{i, k, j\}$.

Appendix B: B-region solution at intermediate distances

If the wave function in eq. (36) is a solution to eq. (35) we must have that

$$B_+^{II}e^{i\sqrt{\tilde{\lambda}}\pi/3} + \frac{4}{i\sqrt{3\tilde{\lambda}}}B_-^{II} = \frac{2A_{II}}{\sqrt{3\tilde{\lambda}}}e^{i\sqrt{\tilde{\lambda}}\pi/6} \tag{1}$$

$$\frac{4}{i\sqrt{3\tilde{\lambda}}}B_+^{II}e^{i\sqrt{\tilde{\lambda}}\pi/3} - B_-^{II} = -\frac{2A_{II}}{\sqrt{3\tilde{\lambda}}}e^{i\sqrt{\tilde{\lambda}}\pi/6} \tag{2}$$

$$((\kappa_B^{(k)})^2 + v_0\rho^2 + \tilde{\lambda})B_+^{(k)}e^{\kappa_B^{(k)}\pi/3} + \frac{4}{\sqrt{3}}\frac{v_0\rho^2}{\kappa_B^{(k)}}B_-^{(k)} = 0 \tag{3}$$

$$\frac{4}{\sqrt{3}}\frac{v_0\rho^2}{\kappa_B^{(k)}}B_+^{(k)}e^{\kappa_B^{(k)}\pi/3} - ((\kappa_B^{(k)})^2 + v_0\rho^2 + \tilde{\lambda})B_-^{(k)} = 0 \tag{4}$$

for each of the values of k=1,2,3 and furthermore

$$\begin{gathered}\frac{A_{II}}{2\sqrt{\tilde{\lambda}}}\left(e^{i\sqrt{\tilde{\lambda}}(\alpha_0 - \pi/2)} + e^{-i\sqrt{\tilde{\lambda}}(\alpha_0 - \pi/2)}\right) \\ + \frac{1}{i\sqrt{\tilde{\lambda}}}\left(B_+^{II}e^{i\sqrt{\tilde{\lambda}}(2\pi/3 - \alpha_0)} - B_-^{II}e^{-i\sqrt{\tilde{\lambda}}(2\pi/3 - \alpha_0)}\right) \\ + \sum_{k=1}^{3}\left[\frac{1}{\kappa_B^{(k)}}\left(B_+^{(k)}e^{\kappa_B^{(k)}(2\pi/3 - \alpha_0)} - B_-^{(k)}e^{-\kappa_B^{(k)}(2\pi/3 - \alpha_0)}\right)\right] + \int_{2\pi/3 - \alpha_0}^{\alpha_0}\phi_C(\rho, \alpha')d\alpha' = 0.\end{gathered} \tag{5}$$

Eqs. (B1) and (B2) have for $\lambda \neq 16/3$ the unique solution

$$B_+^{II} e^{i\sqrt{\tilde{\lambda}}\pi/3} = \frac{1 - \frac{4}{i\sqrt{3\tilde{\lambda}}}}{1 - \frac{16}{3\tilde{\lambda}}} \; \frac{2A_{II}}{\sqrt{3\tilde{\lambda}}} e^{i\sqrt{\tilde{\lambda}}\pi/6} \tag{6}$$

$$B_-^{II} = \frac{1 + \frac{4}{i\sqrt{3\tilde{\lambda}}}}{1 - \frac{16}{3\tilde{\lambda}}} \; \frac{2A_{II}}{\sqrt{3\tilde{\lambda}}} e^{i\sqrt{\tilde{\lambda}}\pi/6} \; . \tag{7}$$

For $\tilde{\lambda} = 16/3$ the two equations determining $B_\pm^{II}$ are identical and infinitely many sets of coefficients exist. The constraint, in addition to eq. (B5), on $B_\pm^{II}$ are then

$$B_+^{II} e^{i\sqrt{\tilde{\lambda}}\pi/3} + \frac{4}{i\sqrt{3\tilde{\lambda}}} B_-^{II} = \frac{2A_{II}}{\sqrt{3\tilde{\lambda}}} e^{i\sqrt{\tilde{\lambda}}\pi/6} \; . \tag{8}$$

Eqs. (B3) and (B4) only have non-trivial solutions when

$$(\kappa_B^{(k)})^2 + v_0\rho^2 + \tilde{\lambda} = \pm \frac{4i}{\sqrt{3}} \frac{v_0\rho^2}{\kappa_B^{(k)}} \; , \tag{9}$$

which for each of the possible signs is a cubic equation with the three complex solutions $\kappa_B^{(k)}$ where k=1,2,3. The corresponding coefficients are then related by

$$\pm \, iB_-^{(k)} = B_+^{(k)} e^{\kappa_B^{(k)}\pi/3} \; , \quad k = 1, 2, 3. \tag{10}$$

The solutions corresponding to the different signs are related by an interchange of the $B_+^{(k)}$ and $B_-^{(k)}$ terms in eq. (36).

With the expressions in eqs. (B6), (B7) and (B10) for the coefficients we can rewrite the link to the C-region from eq. (B5) as

$$\begin{aligned} \int_{2\pi/3-\alpha_0}^{\alpha_0} \phi_C(\rho, \alpha') d\alpha' = & - \sum_{k=1}^{3} \left[\frac{B_-^{(k)}}{\kappa_B^{(k)}} \left(\pm i e^{\kappa_B^{(k)}(\pi/3-\alpha_0)} - e^{-\kappa_B^{(k)}(2\pi/3-\alpha_0)} \right) \right] \\ & - \frac{A_{II}}{2\sqrt{\tilde{\lambda}}} \; \frac{1}{1 - \frac{16}{3\tilde{\lambda}}} \left(e^{i\sqrt{\tilde{\lambda}}(\pi/2-\alpha_0)} (1 + \frac{4}{i\sqrt{3\tilde{\lambda}}}) + e^{-i\sqrt{\tilde{\lambda}}(\pi/2-\alpha_0)} (1 - \frac{4}{i\sqrt{3\tilde{\lambda}}}) \right) \; . \end{aligned} \tag{11}$$

Using the wave function ϕ_C explicitly together with the expressions for the C-coefficients from appendix C, we calculate the integral in eq. (B5) and arrive at

$$\int_{2\pi/3-\alpha_0}^{\alpha_0} \phi_C(\rho, \alpha') d\alpha' = \sum_{k=1}^{3} \left[\frac{\mp i A^{(k)} \sqrt{3}}{\kappa_{AC}^{(k)}} \left(e^{\kappa_{AC}^{(k)}(\alpha_0 - \pi/3)} - e^{-\kappa_{AC}^{(k)}(\alpha_0 - \pi/3)} \right) \right] \; , \tag{12}$$

where the summation excluded $k = 0$, since this contribution vanishes. Combined with eq. (B11) this gives one constraint between the coefficients A_{II}, $B_-^{(k)}$ and $A^{(k)}$ for k=1,2,3.

Appendix C: A- and C-region solutions at intermediate distances

If the wave functions in eqs. (37) and (38) are solutions to eqs. (33) and (34) we must have that

$$((\kappa_{AC}^{(k)})^2 + v_0\rho^2 + \tilde{\lambda}) A^{(k)} + \frac{4}{\sqrt{3}} \frac{v_0\rho^2}{\kappa_{AC}^{(k)}} (C_+^{(k)} e^{\kappa_{AC}^{(k)}\pi/3} + C_-^{(k)} e^{-\kappa_{AC}^{(k)}\pi/3}) = 0 \tag{1}$$

$$((\kappa_{AC}^{(k)})^2 + v_0\rho^2 + \tilde{\lambda}) C_+^{(k)} e^{\kappa_{AC}^{(k)}\pi/3} - \frac{4}{\sqrt{3}} \frac{v_0\rho^2}{\kappa_{AC}^{(k)}} (A^{(k)} + C_-^{(k)} e^{-\kappa_{AC}^{(k)}\pi/3}) = 0 \tag{2}$$

$$((\kappa_{AC}^{(k)})^2 + v_0\rho^2 + \tilde{\lambda})C_-^{(k)} e^{-\kappa_{AC}^{(k)}\pi/3} - \frac{4}{\sqrt{3}}\frac{v_0\rho^2}{\kappa_{AC}^{(k)}}(A^{(k)} + C_+^{(k)} e^{\kappa_{AC}^{(k)}\pi/3}) = 0 \tag{3}$$

for each of the values of k=0,1,2,3 and furthermore

$$\sum_{k=0}^{3}\left[\frac{A^{(k)}}{\kappa_{AC}^{(k)}}\left(e^{\kappa_{AC}^{(k)}(\alpha_0-\pi/3)} + e^{-\kappa_{AC}^{(k)}(\alpha_0-\pi/3)}\right)\right.$$
$$\left. -\frac{1}{\kappa_{AC}^{(k)}}\left(C_+^{(k)} e^{\kappa_{AC}^{(k)}(2\pi/3-\alpha_0)} - C_-^{(k)} e^{-\kappa_{AC}^{(k)}(2\pi/3-\alpha_0)}\right)\right] + \int_{\alpha_0-\pi/3}^{2\pi/3-\alpha_0}\phi_B(\rho,\alpha')d\alpha' = 0\,. \tag{4}$$

Eqs. (C1), (C2) and (C3) only have non-trivial solutions when the determinant vanishes for this linear system of equations. This is equivalent to either

$$(\kappa_{AC}^{(k)})^2 + v_0\rho^2 + \tilde{\lambda} = 0 \tag{5}$$

or instead, when eq. (C5) is false, that

$$(\kappa_{AC}^{(k)})^2 + v_0\rho^2 + \tilde{\lambda} = \pm i\frac{4v_0\rho^2}{\kappa_{AC}^{(k)}}\,, \tag{6}$$

The first of these equations in eq. (C5) is a second order equation with two solutions for $\kappa_{AC}^{(k)}$. They only differ by a sign in κ and they therefore correspond to the same wave function and count as one solution here labeled by $k = 0$. The corresponding coefficients are then related by

$$C_+^{(0)} e^{\kappa_{AC}^{(0)}\pi/3} = -C_-^{(0)} e^{-\kappa_{AC}^{(0)}\pi/3} = A^{(0)} \tag{7}$$

The expression in eq. (C6) is for each of the possible signs a cubic equation with three complex solutions $\kappa_{AC}^{(k)}$ here labeled by k=1,2,3. The corresponding coefficients are given by

$$C_+^{(k)} e^{\kappa_{AC}^{(k)}\pi/3} = \frac{A^{(k)}}{2}(1 \mp i\sqrt{3})\ \ ,\ \ C_-^{(k)} e^{-\kappa_{AC}^{(k)}\pi/3} = -\frac{A^{(k)}}{2}(1 \pm i\sqrt{3}) \tag{8}$$

for the values of k=1,2,3. The solutions corresponding to the different signs are related by an interchange of the $C_+^{(k)}$ and $C_-^{(k)}$ terms in eq. (38).

With the expressions in eqs. (C7) and (C8) for the coefficients we can rewrite the link to the B-region from eq. (C4) as

$$\int_{\alpha_0-\pi/3}^{2\pi/3-\alpha_0}\phi_B(\rho,\alpha')d\alpha' \tag{9}$$
$$= -\sum_{k=1}^{3}\left[\frac{3A^{(k)}}{2\kappa_{AC}^{(k)}}\left((1 \mp i/\sqrt{3})e^{\kappa_{AC}^{(k)}(\alpha_0-\pi/3)} + (1 \pm i/\sqrt{3})e^{-\kappa_{AC}^{(k)}(\alpha_0-\pi/3)}\right)\right]\,.$$

Here the summation does not include $k = 0$, since this contribution vanishes. Using the wave function ϕ_B explicitly together with the expressions for the B-coefficients from appendix B, we calculate the integral in eq. (C9) and arrive at

$$\int_{\alpha_0-\pi/3}^{2\pi/3-\alpha_0}\phi_B(\rho,\alpha')d\alpha' = -\sum_{k=1}^{3}\left(\frac{B_-^{(k)}(1\pm i)}{\kappa_B^{(k)}}(e^{\kappa_B^{(k)}(\alpha_0-2\pi/3)} - e^{-\kappa_B^{(k)}(\alpha_0-\pi/3)})\right)$$
$$+\left(e^{i\sqrt{\tilde{\lambda}}(\alpha_0-2\pi/3)} - e^{-i\sqrt{\tilde{\lambda}}(\alpha_0-\pi/3)}\right)\frac{4iA_{II}}{\tilde{\lambda}\sqrt{3}}\ \ \frac{1}{1-\frac{16}{3\tilde{\lambda}}}e^{i\sqrt{\tilde{\lambda}}\,\pi/6)}\,, \tag{10}$$

which combined with eq. (C9) gives one constraint between the coefficients A_{II}, $B_-^{(k)}$ and $A^{(k)}$ for k=1,2,3.

Appendix D: Properties of the asymmetric solutions

We have not found a general rigorous proof for the claim that three real solutions exist for $n \geq 3$. However, the numerical computations all unanimously support the theorem and several very different limiting cases shall be discussed below. The intermediate cases are probably similar and with a determined effort possible to prove.

When n increases d_i^2 approaches zero and

$$S_0 \to -v_0^{(1)}v_0^{(2)}v_0^{(3)}\ ,\ S_1 \to v_0^{(1)}v_0^{(2)} + v_0^{(1)}v_0^{(3)} + v_0^{(2)}v_0^{(3)}\ ,\ S_2 \to -v_0^{(1)} - v_0^{(2)} - v_0^{(3)} \tag{1}$$

with the solution $\epsilon = v_0^{(i)}$ to eq. (83).

In general the cubic equation in eq. (83) has three real solutions if

$$\left(\frac{1}{3}S_1 - \frac{1}{9}S_2^2\right)^3 + \left(-\frac{1}{6}S_1S_2 + \frac{1}{2}S_0 + \frac{1}{27}S_2^3\right)^2 \leq 0\ , \tag{2}$$

which can be rewritten as

$$\frac{1}{27}S_1^3 - \frac{1}{108}S_1^2S_2^2 + \frac{1}{4}S_0^2 + \frac{1}{27}S_0S_2^3 - \frac{1}{6}S_0S_1S_2 \leq 0\ . \tag{3}$$

When one of the potentials, for example $v_0^{(3)}$, is very small or vanishes we get in this limit

$$S_0 \to 0\ ,\ S_1 \to v_0^{(1)}v_0^{(2)}(1-d_3^2)\ ,\ S_2 \to -v_0^{(1)} - v_0^{(2)} \tag{4}$$

and the condition in eq. (D3) becomes

$$\frac{1}{27}S_1^3 - \frac{1}{108}S_1^2S_2^2 = -\frac{1}{27}\left(\frac{1}{4}(v_0^{(1)} - v_0^{(2)})^2 + v_0^{(1)}v_0^{(2)}d_3^2\right) \leq -\frac{1}{27}(v_0^{(1)})^2(1-d_3^2) \leq 0\ . \tag{5}$$

When all $v_0^{(i)}$ are equal the condition in eq. (D3) reduces to

$$\frac{1}{27}x^3 - \frac{1}{12}x^2 + \frac{1}{4}y^2 - y - \frac{1}{2}xy \leq 0\ , \tag{6}$$

where the overall factor $(v_0^{(i)})^6$ has been removed and the new variables x and y are defined as

$$x = 3 - d_1^2 - d_2^2 - d_3^2\ ,\ \ y = 2d_1d_2d_3 + d_1^2 + d_2^2 + d_3^2 - 1\ . \tag{7}$$

Direct computation of the quantities reformulates eq. (D6) into

$$-\frac{1}{27}(d_1^2 + d_2^2 + d_3^2)^3 + (d_1d_2d_3)^2 \leq 0\ , \tag{8}$$

which is fulfilled for any set of d_i.

Appendix E: Eigenvalue equation at large distances for three different particles

The matching conditions at $\alpha_0^{(1)}, \alpha_0^{(2)}$ and $\alpha_0^{(3)}$ at large distances for the asymmetric case provide the eigenvalue equation. This is found by equating the functions in eqs. (87) and (88) and their first derivatives, i.e.

$$\left[a_i \sin\left((\alpha_0^{(i)} - \pi/2)\sqrt{\tilde{\lambda}}\right) - c_i \sin\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right)\right] = b_i \sin(\alpha_0^{(i)}\kappa_i) \tag{1}$$

$$\left[a_i \cos\left((\alpha_0^{(i)} - \pi/2)\sqrt{\tilde{\lambda}}\right) - c_i \cos\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right)\right] \sqrt{\tilde{\lambda}} = b_i\kappa_i \cos(\alpha_0^{(i)}\kappa_i) \tag{2}$$

and then eliminating b_i resulting in

$$\begin{aligned}&\left[a_i \sin\left((\alpha_0^{(i)} - \pi/2)\sqrt{\tilde{\lambda}}\right) - c_i \sin\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right)\right] \kappa_i \cos(\alpha_0^{(i)}\kappa_i) = \\ &\left[a_i \cos\left((\alpha_0^{(i)} - \pi/2)\sqrt{\tilde{\lambda}}\right) - c_i \cos\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right)\right] \sqrt{\tilde{\lambda}} \sin(\alpha_0^{(i)}\kappa_i)\ .\end{aligned} \tag{3}$$

Furthermore eliminating c_i by use of eqs. (89)-(90) then provide 3 linear equations in a_i. They only have non-trivial solutions when the corresponding determinant, $D = \det\{d_{ik}\}$ vanishes. The matrix elements are

$$d_{ii} = \kappa_i \sin\left((\alpha_0^{(i)} - \pi/2)\sqrt{\tilde{\lambda}}\right) \cos(\alpha_0^{(i)}\kappa_i) - \sqrt{\tilde{\lambda}} \cos\left((\alpha_0^{(i)} - \pi/2)\sqrt{\tilde{\lambda}}\right) \sin(\alpha_0^{(i)}\kappa_i) \tag{4}$$

$$d_{ik} = \frac{A_i f_j}{F} \ , \ \text{for} \ \ i \neq k \ , \tag{5}$$

where f_j is defined in eq. (90) and

$$A_i = \tfrac{2F}{\sqrt{\tilde{\lambda}}} \left[\kappa_i \sin\left(\alpha_0^{(i)} \sqrt{\tilde{\lambda}}\right) \cos(\alpha_0^{(i)} \kappa_i) - \sqrt{\tilde{\lambda}} \cos\left(\alpha_0^{(i)} \sqrt{\tilde{\lambda}}\right) \sin(\alpha_0^{(i)} \kappa_i) \right] \tag{6}$$

By further defining

$$B_i = \frac{d_{ii} f_i^2}{F^2} \tag{7}$$

the determinant can then be written as

$$D = B_1 B_2 B_3 + 2A_1 A_2 A_3 - B_1 A_2 A_3 - A_1 A_2 B_3 - A_1 B_2 A_3 \ . \tag{8}$$

The eigenvalues $\tilde{\lambda}$ are then determined by $D = 0$.

Appendix F: Solutions to the case of two identical spin-1/2 particles

The solutions to the Faddeev equations in eqs. (106) and (107) are first found independently for each of the four regions of fig.4. We shall use the notation $\alpha_{ff} = \alpha_1$ and $\alpha_{fc} = \alpha_2$. In region II , where all potentials are zero we have

$$\phi_s^{(i)}(\alpha_i) = A_i \sin\left((\alpha_i - \pi/2) \sqrt{\tilde{\lambda}} \right) \ , \tag{1}$$

where $(i, s) = (1, 0), (2, s_c - 1/2), (3, s_c + 1/2)$ and eq. (105) relate these to the remaining spin components of $\phi^{(3)}$ and $\phi^{(2)}$.

In region B of fig.4, where only one of the potentials is identically zero, we have instead

$$\phi_0^{(1)}(\alpha_{ff}) = b_1 \sin(\alpha_{ff} \kappa_0) + a_0 \sin(\alpha_{ff} \sqrt{\tilde{\lambda}}) \tag{2}$$

$$\phi_s^{(i)}(\alpha_{fc}) = b_i \sin\left((\alpha_{fc} - \pi/2) \sqrt{\tilde{\lambda}} \right) \ , \tag{3}$$

where $(i, s) = (2, s_c - 1/2), (3, s_c + 1/2)$, $\kappa_0 = \sqrt{v_0^{(ff)} \rho^2 + \tilde{\lambda}(\rho)}$ and

$$a_0 = -\frac{4f}{\sqrt{\tilde{\lambda}}} (C^{12}_{0, s_c - 1/2} A_2 - C^{12}_{0, s_c + 1/2} A_3) \ , \tag{4}$$

where we in analogy to eq. (90) define

$$f = \frac{\sin\left((\varphi - \pi/2) \sqrt{\tilde{\lambda}} \right)}{\sin(2\varphi)} \ , \ \tilde{f} = \frac{\sin\left((\tilde{\varphi} - \pi/2) \sqrt{\tilde{\lambda}} \right)}{\sin(2\tilde{\varphi})} \ . \tag{5}$$

In region C of fig.4, where the other potential is identically zero, we have

$$\phi_0^{(1)}(\alpha_{ff}) = c_1 \sin\left((\alpha_{ff} - \pi/2) \sqrt{\tilde{\lambda}} \right) \tag{6}$$

$$\phi_s^{(i)}(\alpha_{fc}) = c_i \sin(\alpha_{fc} \kappa_s) + a_s \sin(\alpha_{fc} \sqrt{\tilde{\lambda}}) \ , \tag{7}$$

where $(i, s) = (2, s_c - 1/2), (3, s_c + 1/2)$, $\kappa_{s_c \pm 1/2} = \sqrt{v_0^{(fc\pm)} \rho^2 + \tilde{\lambda}(\rho)}$ and

$$a_{s_c - 1/2} = -\frac{2}{\sqrt{\tilde{\lambda}}} \left[C^{12}_{0, s_c - 1/2} A_1 f + (C^{23}_{s_c - 1/2, s_c - 1/2} A_2 + C^{23}_{s_c - 1/2, s_c + 1/2} A_3) \tilde{f} \right] \tag{8}$$

$$a_{s_c + 1/2} = -\frac{2}{\sqrt{\tilde{\lambda}}} \left[c^{12}_{0, s_c + 1/2} A_1 f + (-C^{23}_{s_c - 1/2, s_c + 1/2} A_2 + C^{23}_{s_c + 1/2, s_c + 1/2} A_3) \tilde{f} \right] \ . \tag{9}$$

In region D of fig.4, where both potentials are finite, we have

$$\phi_s^{(i)}(\alpha_i) = d_i \sin(\alpha_i \kappa_s) + a_s \sin(\alpha_i \sqrt{\tilde{\lambda}}) \ , \tag{10}$$

where $(i,s) = (1,0), (2, s_c - 1/2), (3, s_c + 1/2)$.

Imposing continuity of the functions and their first derivatives at the boundaries between the different regions of fig.4, the constants must be related by

$$c_1 = A_1 \ , \ b_2 = A_2 \ , \ b_3 = A_3 \ , \ d_1 = b_1 \ , \ d_2 = c_2 \ , \ d_3 = c_3 \ , \tag{11}$$

and the remaining six constants obey the following set of six linear equations:

$$\begin{aligned}
A_1 \sin\left((\alpha_0^{(ff)} - \tfrac{\pi}{2})\sqrt{\tilde{\lambda}}\right) &= b_1 \sin(\alpha_0^{(ff)}\kappa_0) + a_0 \sin(\alpha_0^{(ff)}\sqrt{\tilde{\lambda}}) \\
A_1\sqrt{\tilde{\lambda}} \cos\left((\alpha_0^{(ff)} - \tfrac{\pi}{2})\sqrt{\tilde{\lambda}}\right) &= b_1\kappa_0 \cos(\alpha_0^{(ff)}\kappa_0) + a_0\sqrt{\tilde{\lambda}} \cos(\alpha_0^{(ff)}\sqrt{\tilde{\lambda}}) \\
A_2 \sin\left((\alpha_0^{(fc)} - \tfrac{\pi}{2})\sqrt{\tilde{\lambda}}\right) &= c_2 \sin(\alpha_0^{(fc)}\kappa_{s_c-1/2}) + a_{s_c-1/2} \sin(\alpha_0^{(fc)}\sqrt{\tilde{\lambda}}) \\
A_2\sqrt{\tilde{\lambda}} \cos\left((\alpha_0^{(fc)} - \tfrac{\pi}{2})\sqrt{\tilde{\lambda}}\right) &= c_2\kappa_{s_c-1/2} \cos(\alpha_0^{(fc)}\kappa_{s_c-1/2}) - a_{s_c-1/2}\sqrt{\tilde{\lambda}} \cos(\alpha_0^{(fc)}\sqrt{\tilde{\lambda}}) \\
A_3 \sin\left((\alpha_0^{(fc)} - \tfrac{\pi}{2})\sqrt{\tilde{\lambda}}\right) &= c_3 \sin(\alpha_0^{(fc)}\kappa_{s_c+1/2}) + a_{s_c+1/2} \sin(\alpha_0^{(fc)}\sqrt{\tilde{\lambda}}) \\
A_3\sqrt{\tilde{\lambda}} \cos\left((\alpha_0^{(fc)} - \tfrac{\pi}{2})\sqrt{\tilde{\lambda}}\right) &= c_3\kappa_{s_c+1/2} \cos(\alpha_0^{(fc)}\kappa_{s_c+1/2}) + a_{s_c+1/2}\sqrt{\tilde{\lambda}} \cos(\alpha_0^{(fc)}\sqrt{\tilde{\lambda}}) \ .
\end{aligned} \tag{12}$$

Since a_s only depend on A_i we can easily eliminate b_1, c_2, c_3 from the equations in eq. (F12). This leaves 3 homogeneous linear equations in A_1, A_2, A_3 with a corresponding determinant $D = \det\{d_{ik}\}$. The matrix elements are given by

$$d_{22} = D_2 + F_{23} C^{23}_{s_c-1/2, s_c-1/2} \ , \ d_{33} = D_3 + F_{32} C^{23}_{s_c+1/2, s_c+1/2} \ , \tag{13}$$

$$d_{11} = D_1 \ , \ d_{ik} = F_{ik} C^{ik}_{s_i, s_k} \ , \tag{14}$$

where $s_1 = 0, s_2 = s_c - 1/2, s_3 = s_c + 1/2$, the spin overlap functions are defined in eqs. (15) and

$$D_i = \kappa_s \sin\left((\alpha_0^{(i)} - \frac{\pi}{2})\sqrt{\tilde{\lambda}}\right) \cos(\alpha_0^{(i)}\kappa_s) - \sqrt{\tilde{\lambda}} \cos\left((\alpha_0^{(i)} - \frac{\pi}{2})\sqrt{\tilde{\lambda}}\right) \sin(\alpha_0^{(i)}\kappa_s) \ , \tag{15}$$

$$F_{ik} = \left[\kappa_s \sin\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right) \cos(\alpha_0^{(i)}\kappa_s) - \sqrt{\tilde{\lambda}} \cos\left(\alpha_0^{(i)}\sqrt{\tilde{\lambda}}\right) \sin(\alpha_0^{(i)}\kappa_s)\right] \frac{2(1+\delta_{i,1})}{\sqrt{\tilde{\lambda}}} f_j \ , \tag{16}$$

where f_j is defined in eq. (90).

The values of $\tilde{\lambda}$ are as usual determined from $D = 0$.

Simplified expressions can be obtained in the limit of very large distances where

$$F_{ik} \approx -2(1+\delta_{i,1})\mu_{jk} \sqrt{v_0^{(i)}} a_{scat}^{(i)} \cos(X_i \sqrt{v_0^{(i)}}) f_j \ , \tag{17}$$

$$D_i \approx -\cos(X_i \sqrt{v_0^{(i)}}) \sqrt{v_0^{(i)}} \left(\rho \sin(\frac{\pi}{2}\sqrt{\tilde{\lambda}}) + a_{scat}^{(i)} \mu_{jk} \sqrt{\tilde{\lambda}} \cos(\frac{\pi}{2}\sqrt{\tilde{\lambda}})\right) \ . \tag{18}$$

Here $i = 1, 2, 3$ corresponds to $ff, fc-, fc+$, respectively. Then $v_0^{(i)}$ is defined in eqs. (114) and (115) and the scattering lengths, $a_{scat}^{(ff)}$, $a_{scat}^{(fc+)}$, $a_{scat}^{(fc-)}$, for the two-body systems are defined in analogy with eq. (95) in terms of radii, potentials and reduced masses.

When $\tilde{\lambda}/\rho^2$ remains finite the dominating terms in the determinant are D_i. Thus D is approximately diagonal with the matrixelemnts D_i given in eq. (F15). The eigenvalues in the limit of large ρ then again corresponds to the two-body bound states obtained by using eq. (F15) and solving $D_i = 0$.

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